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Accurate modelling of pulse transformation by adjustable-in-time medium parameters

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Abstract. A possibility of a strong change of an electromagnetic signal by a short sequence of time cycles of pulses that modulate the medium parameters is shown. The backward wave is demonstrated to be an inevitable result of the medium time change. Dependence of the relation between backward and forward waves on the parameters of the medium modulation is investigated. The finite statistical complexity of the electromagnetic signal transformed by a finite sequence of modulating cycles is calculated. Increase of the complexity with the number of cycles is shown.

Key words: electromagnetic transients, integral equations in time domain, time-varying medium

1. Introduction

Interactions between optical pulses and semiconductor active media in waveguides are of significant importance in optical communication technology (Chi *et al.* 2001). Optical frequency conversion represents an important on-going issue in optics and optical engineering and has received considerable attention lately (Jeong and Lee 2001). Recently time-domain techniques for solving such electromagnetic problems have been actively discussed in the literature, mainly owing to their superiority in solving wide-band problems in comparison to frequency-domain methods (Shifman and Leviatan 2001).

Parametric phenomena in active media have long been studied in connection with the generation and amplification of electromagnetic waves through the charges interaction with a spatial periodic medium (Fainberg and Khizhnjak 1957; Bekefi 1986), or by the time modulation of the medium parameters (Morgenthaler 1958; Ostrovsky and Stepanov 1971; Averkov *et al.* 1980; Stolyarov 1983; Harfoush and Taflove 1991). Other goal of investigations of such transient electromagnetic phenomena is a problem of electromagnetic signal controlling by a temporal adjustment of the medium parameters, especially the semiconductor medium, that is important in developing optoelectronic systems (Wiesenfeld 1998). One of the possible ways of such an adjustment consists of a change of medium parameters by a finite packet of identical cycles of the change.

It is known that a single abrupt change of medium parameters leads to changing of the wave frequency and amplitude or the pulse shape. To construct the desired behaviour of the field one has to use a complex modulation of medium parameters. However, a sinusoidal disturbance of the permittivity or conductivity is mostly used. Even in this case, not to mention more sophisticated kinds of modulation, the interaction is complex and its investigation can be made only approximately, especially when the period of modulation cycles is comparable with the period of the affected field. The complexity of the problem arises when only a few cycles of the modulation are considered as in ultrafast optics.

However, modelling of these interactions can reveal major features of the phenomena even using a simple law of the change of the medium parameters like a finite packet of pulses of a simple form. It has been shown in (Nerukh 1999) that in the case of the sequence of the rectangular pulses the amplitude of the harmonic plane wave can have monotonic as well as irregular behaviour with increasing the number of pulses.

In this paper the influence of the medium parameters altered by the law of a finite sequence of rectangular cycles on an electromagnetic pulse is considered. Investigation is based on the Volterra integral equation method that allows considering problems with arbitrary primary signal.

In the statement of the problem the values of the permittivity and the conductivity jumps do not have any limitations nor do the durations of the intervals of the parameters change. It is shown that the influence of one cycle of the medium modulation on the signal leads to splitting the latter into two new signals with the same envelope. Next cycles of the medium modulation make the picture very complex, its behaviour changes in a complex way from one modulation cycle to another, and it depends strongly on the number of the modulation cycles and on the phase of the medium modulation cycle at which the modulation is stopped. Exact expressions for transformation of a plane electromagnetic wave by a finite packet of cycles of the medium parameters change are obtained. This allows analysing relationships between backward and forward secondary waves on the disturbance intervals as well as on the inactivity ones.

It is shown that in the case of non-dissipative medium one can derive the exact expressions for the transformed field induced by an arbitrary initial signal. Pulse shaping in such a non-dissipative medium for the primary Gaussian pulse is investigated in detail on both disturbance and inactivity intervals of an arbitrary modulation cycle. Dependence of the pulse transformation and its complexity on the number of cycles is calculated. The complexity calculation is done in the framework of Crutchfield's 'computational mechanics' approach.

2. Evolution equations for electric field

We consider an unbounded dielectric dissipative medium where electromagnetic field $E_0(t,x)$ exists. Beginning from the zero moment of time the medium permittivity and conductivity change under the action of the external forces as a finite packet of repeating cycles. Thus, the permittivity and the conductivity of the medium receive constant magnitudes ε_1 and σ on the disturbance intervals of the cycles and they have constant magnitudes ε and $\sigma = 0$ on the inactivity intervals of the cycles.

The problem is described by the Volterra integral equation approach (Nerukh and Khizhnyak 1991; Nerukh *et al.* 1998, 2001), according to which a lateral-to-x-axes component of the electric field vector on the medium disturbance interval on the nth cycle, $(n-1)T < t < T_1 + (n-1)T$, $n = 1, \ldots, N$ where T is the duration of the cycle of the parameters change, T_1 is the duration of the disturbance interval of each cycle, is determined by the operator formula (Nerukh 1999)

$$E_n = F_{n-1} + RF_{n-1}, \quad n \geq 1 \tag{1}$$

where R is the problem resolvent operator (see Equation (16) in Appendix A).

The free term in Equation (1) is determined by the field magnitudes on the previous cycles

$$F_{n-1}(t,x) = F_0(t,x) - \frac{1}{2a^2}\left\{2b + (1-a^2)\frac{\partial}{\partial t}\right\}\sum_{k=1}^{n-1}\int_{kT}^{kT+T_1} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x'\, \delta(v(t-t') - |x-x'|)E_k(t',x'), \quad n \geq 2 \tag{2}$$

In this equation $a = \sqrt{\varepsilon/\varepsilon_1}$ and $b = \sigma/(\varepsilon_0\varepsilon_1)$ are normalized medium parameters, ε_0 is the vacuum permittivity, $v = c/\sqrt{\varepsilon}$ and $v_1 = c/\sqrt{\varepsilon_1}$ are the phase velocities, c is the light velocity in vacuum, and $\delta(t)$ is the Dirack delta-function. The values $(\varepsilon_1 - \varepsilon)$ and σ_1 do not have any limitations, nor do the durations T and T_1.

The free term with zero index is equal to the initial field $F_0(t,x) = E_0(t,x)$.

On the inactivity interval of the nth cycle, $T_1 + (n-1)T < t < nT$, $n = 1, \ldots, N$, the field equals to the free term of the equation, $E_n = F_n$. An example for the time intervals when the packet consists of three cycles, $N = 3$, is shown in the Fig. 1.

3. Signal transformation by one time jump of medium parameters

First, we consider the transformation of the primary field $E_0(t,x) = F_0(t,x) = f(x \pm vt)$ caused by a single time jump of the permittivity and

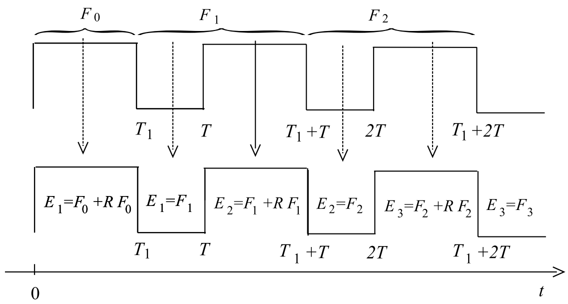


Fig. 1. Time intervals for the definition of the equations.

conductivity at zero moment when they change to the values ε_1 and σ ($a = const, b = const$). Here, $f(x)$ is an arbitrary function describing the field before the zero moment. The formula (1) with $n = 1$ yields the transformed field (Nerukh 1999):

$$E_1(t,x) = \frac{a}{2}\exp(-bt)[(a \mp 1)f(x - v_1 t) + (a \pm 1)f(x + v_1 t)] + \Phi(t,x). \quad (3)$$

where

$$\Phi(t,x) = \frac{ab}{2}\exp(-bt)\int_0^t \Big\{ \pm\Big[f\Big(x + v_1\sqrt{t^2 - z^2}\Big) - f\Big(x - v_1\sqrt{t^2 - z^2}\Big)\Big] I_1(bz) + a\Big[f\Big(x + v_1\sqrt{t^2 - z^2}\Big) + f\Big(x - v_1\sqrt{t^2 - z^2}\Big)\Big]\frac{tI_1(bz) - zI_0(bz)}{\sqrt{t^2 - z^2}}\Big\}\,\mathrm{d}z \quad (4)$$

and $I_n(z)$ is the modified Bessel function.

The terms in the square brackets in Equation (3) describe two waves that travel in opposite directions with new phase velocity and may be thought of as splitting of the primary wave. They give a discrete part of the transformed signal while the last term $\Phi(t,x)$ is the continuous superposition of the waves that is formed only when the medium conductivity is non-zero ($b \neq 0$) and it is absent in non-dissipative medium.

If the primary field is a plane harmonic wave, that is $f = \exp[\mathrm{i}(\mp\omega t - kx)]$, the continuous superposition can be calculated explicitly (Nerukh 1999) and the transformed field consists of two waves

$$E_1(t,x) = \frac{a^2}{2}\exp(-bt - \mathrm{i}kx)\left[\left(1 + \frac{\mp\omega + \mathrm{i}b}{\Omega}\right)\exp(\mathrm{i}t\Omega) + \left(1 - \frac{\mp\omega + \mathrm{i}b}{\Omega}\right)\exp(-\mathrm{i}t\Omega)\right] \quad (5)$$

with new amplitudes and new frequencies $\Omega = \sqrt{\omega_1^2 - b^2}$ where $\omega_1 = a\omega$.

4. Transformation of a plane electromagnetic wave by a finite packet of modulation cycles

The same picture we observe in the case of N cycles of the medium parameters change, namely, the primary harmonic wave $E_0(t,x) = \exp[\mathrm{i}(\omega t - kx)]$ is transformed into two (direct and inverse) waves $E_n = \exp(-bt - \mathrm{i}kx) \times [C_n \exp(\mathrm{i}\Omega t) + D_n \exp(-\mathrm{i}\Omega t)]$ on each disturbance interval of the nth cycle (Nerukh 1999). These waves have the same frequency as in Equation (5) but their amplitudes change in more complex way

$$\begin{aligned} C_n &= \frac{a^2}{2\Omega}\exp((n-1)bT - \mathrm{i}(n-1)\Omega T)[c_{11}A_{n-1}\exp(\mathrm{i}(n-1)\omega T) \\ &\quad + c_{12}B_{n-1}\exp(-\mathrm{i}(n-1)\omega T)], \\ D_n &= \frac{a^2}{2\Omega}\exp((n-1)bT + \mathrm{i}(n-1)\Omega T)[c_{21}A_{n-1}\exp(\mathrm{i}(n-1)\omega T) \\ &\quad + c_{22}B_{n-1}\exp(-\mathrm{i}(n-1)\omega T)], \ n > 1, \end{aligned} \tag{6}$$

where $c_{11} = \Omega + \omega + \mathrm{i}b,\ c_{12} = \Omega - \omega + \mathrm{i}b,\ c_{21} = \Omega - \omega - \mathrm{i}b,\ c_{22} = \Omega + \omega - \mathrm{i}b$ and A_n, B_n are amplitudes of the waves on the inactivity interval of the previous cycle.

On the inactivity interval of the nth cycle the field $E_n = F_n = \exp(-\mathrm{i}kx)[A_n \exp(\mathrm{i}\omega t) + B_n \exp(-\mathrm{i}\omega t)]$ also consists of two waves but with the frequencies of the primary wave and new amplitudes that satisfy the recurrent relations

$$\begin{aligned} A_n &= \frac{1}{2\Omega}\exp(-(b+\mathrm{i}\omega)T_1)[l_{11}A_{n-1} + l_{12}B_{n-1}\exp(-\mathrm{i}2(n-1)\omega T)], \\ B_n &= \frac{-1}{2\Omega}\exp(-(b-\mathrm{i}\omega)T_1)[l_{21}A_{n-1}\exp[(\mathrm{i}2(n-1)\omega T) + l_{22}B_{n-1}], \quad n > 1, \end{aligned} \tag{7}$$

where

$$\begin{aligned} l_{11} &= \mathrm{i}\omega(a^2+1)\sin\Omega T_1 + 2\Omega\cos\Omega T_1, l_{12} = \mathrm{i}\left[\omega(a^2-1) + 2\mathrm{i}b\right]\sin\Omega T_1 \\ l_{21} &= \mathrm{i}\left[\omega(a^2-1) - 2\mathrm{i}b\right]\sin\Omega T_1, l_{22} = \mathrm{i}\omega(a^2+1)\sin\Omega T_1 - 2\Omega\cos\Omega T_1 \end{aligned}$$

So, the influence of one cycle of the medium modulation onto the electromagnetic wave leads to splitting of the latter into two new waves with the new frequencies and amplitudes and travelling in opposite directions with a new phase velocity. The next cycles of medium modulation make the picture of the field very complex. Nevertheless, in the case of a harmonic plane wave the transformed field preserves the structure of the first transformation, so that it consists of two harmonic plane waves with remaining wavenumber, bounding frequency and amplitudes changing in a complex way from one modulation cycle to another. Appearance of inverse waves is inevitable result of the medium time change.

5. Relations between direct and inverse waves

To simplify the notation it is convenient to use the dimensionless parameters $\tilde{T} = \omega T, \tilde{T}_1 = \omega T_1$ and the reduced amplitudes $a_n = A_n \exp(\mathrm{i}n\tilde{T})$, $b_n = B_n \exp(-\mathrm{i}n\tilde{T})$, $c_n = C_n \exp(\mathrm{i}(n-1)q\tilde{T})$, $d_n = D_n \exp(-\mathrm{i}(n-1)q\tilde{T})$ where $q = \sqrt{a^2 - s^2}$, $s = b/\omega$.

The relation between the direct and inverse secondary wave amplitudes following from Equations (6) and (7) are given by the ratios:
on the disturbance intervals:

$$w_N = \frac{d_N}{c_N} = \frac{\{p_2 + (p_1 - p_2)r_{N-1}\}\bar{c}_{21} + p_1 p_2 \bar{c}_{22}}{\{p_2 + (p_1 - p_2)r_{N-1}\}\bar{c}_{11} + p_1 p_2 \bar{c}_{12}}, \tag{8}$$

on the inactivity intervals:

$$p_N = \frac{b_N}{a_N} = \frac{p_1 p_2}{p_2 + (p_1 - p_2)r_N}, \quad N \geq 2, \tag{9}$$

Here,

$$p_1 = -\frac{h}{m}, \quad p_2 = -\frac{h(m + m^*)}{hh^* + m^2}, \quad \bar{c}_{11} = \bar{c}_{22}^* = q + 1 + \mathrm{i}s,$$
$$\bar{c}_{12} = \bar{c}_{21}^* = q - 1 + \mathrm{i}s, \quad a_1 = m, \quad b_1 = -h,$$

$$m = \frac{2q\cos(q\tilde{T}_1) + \mathrm{i}(a^2+1)\sin(q\tilde{T}_1)}{2q}\exp[-s\tilde{T}_1 + \mathrm{i}(\tilde{T} - \tilde{T}_1)],$$

$$h = \mathrm{i}\frac{1}{2q}(a^2 - 1 - \mathrm{i}2s)\sin(q\tilde{T}_1)\exp[-s\tilde{T}_1 - \mathrm{i}(\tilde{T} - \tilde{T}_1)]$$

and symbol (*) denotes the complex conjugate value. The variable r_N is determined by a real sequence

$$r_{N+1} = \frac{4u^2}{4u^2 - r_N}, \quad r_1 = 0.$$

which behaviour and behaviour of the amplitude ratios w_N and p_N are determined by the magnitude of the generalized parameter

$$u = \cos(q\widetilde{T}_1)\cos(\widetilde{T} - \widetilde{T}_1) - \frac{a^2 + 1}{2q}\sin(q\widetilde{T}_1)\sin(\widetilde{T} - \widetilde{T}_1). \quad (10)$$

When $u^2 > 1$ the amplitudes change monotonically with the number of modulation cycles as shown in Fig. 2 for $u = 1.0002$. The behaviour has non-monotone and even irregular character if $u^2 < 1$. Fig. 3 corresponds to $u = 0.98988$ when the monotonic behaviour of the amplitudes ratio breaks and at some cycle the amplitude of the inverse wave relative to the direct one is minimal. In both cases the ratio of the amplitudes on the disturbance and inactivity intervals have similar character. The case when the behaviour of

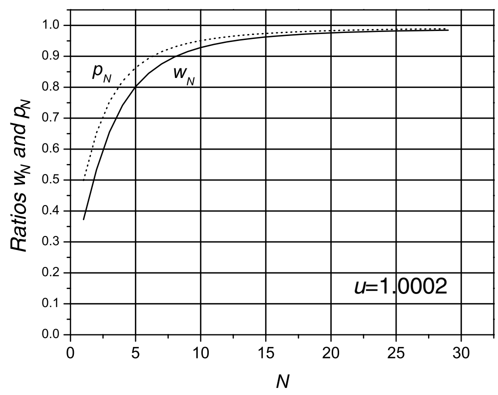


Fig. 2. Relations between direct and inverse secondary wave amplitudes for the parameters of modulation $a = 1.5$, $s = 0.01$, $\widetilde{T}_1 = 3.68$, $\widetilde{T}_2 = 0.52$.

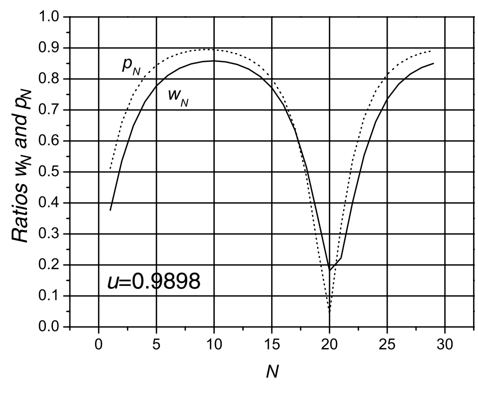


Fig. 3. Relations between direct and inverse secondary wave amplitudes for the parameters of modulation $a = 1.5$, $s = 0.01$, $\widetilde{T}_1 = 3.65$, $\widetilde{T}_2 = 0.52$.

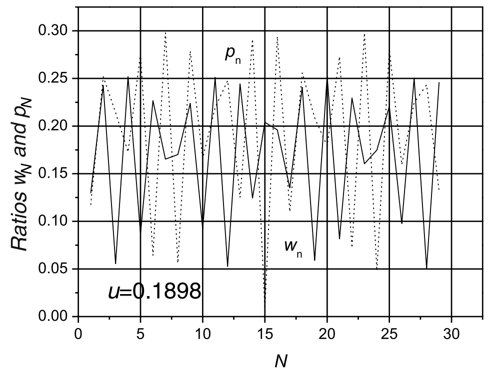


Fig. 4. Relations between direct and inverse secondary wave amplitudes for the parameters of modulation $a = 1.5$, $s = 0.01$, $\widetilde{T}_1 = 0.55$, $\widetilde{T}_2 = 0.52$.

the amplitudes has quasi-chaotic character is shown in Fig. 4. Here $u = 0.18978$ and the ratio of the amplitudes on the disturbance intervals does not correlate with the ratio on the inactivity ones. This allows changing the ratio considerably by small variations of the number of modulation cycles.

6. Pulse shaping in non-dissipative medium

When the conductivity of the medium is equal to zero the exact expressions for the transformed field can be derived explicitly for an arbitrary initial signal $E_0(t,x) = F_0(t,x) = f(x - vt)$ using a simple form for the resolvent, see Equation (18) in Appendix A. The transformed field on the disturbance interval of the nth cycle, $n > 1$, consists of two, forward and backward signals

$$E_n(t,x) = E_n^{(+)}(t - x/v_1) + E_n^{(-)}(t + x/v_1), \tag{11}$$

that are determined by the free term on the previous cycle:

$$\begin{aligned} E_n^{(+)}(t - x/v_1) = \frac{a}{2}\Bigg[&(a+1)F_{n-1}^{(+)}\left(\frac{t - x/v_1}{a} - (a-1)(n-1)T\right) \\ &+(a-1)F_{n-1}^{(-)}\left(-\frac{t - x/v_1}{a} - (a+1)(n-1)T\right)\Bigg], \\ E_n^{(-)}(t + x/v_1) = \frac{a}{2}\Bigg[&(a-1)F_{n-1}^{(+)}\left(-\frac{t + x/v_1}{a} - (a+1)(n-1)T\right) \\ &+(a+1)F_{n-1}^{(-)}\left(\frac{t + x/v_1}{a} + (a-1)(n-1)T\right)\Bigg]. \end{aligned} \tag{12}$$

On the inactivity interval of the nth cycle the field is equal to the free term

$$E_n = F_n(t,x) = F_n^{(+)}(t - x/v) + F_n^{(-)}(t + x/v), \tag{13}$$

where

$$\begin{aligned}
F_n^{(+)}(t - x/v) &= F_0(t - x/v) \\
&\quad - \frac{1}{2a^2}\sum_{k=0}^{n-1}\Bigg[(a+1)\Bigg\{E_{k+1}^{(+)}\left(\frac{t - x/v}{a} + \frac{a-1}{a}(kT + T_1)\right) \\
&\quad - E_{k+1}^{(+)}\left(\frac{t - x/v}{a} + \frac{a-1}{a}kT\right)\Bigg\} \\
&\quad - (a-1)\Bigg\{E_{k+1}^{(-)}\left(-\frac{t - x/v}{a} + \frac{a+1}{a}(kT + T_1)\right) \\
&\quad - E_{k+1}^{(-)}\left(-\frac{t - x/v}{a} + \frac{a+1}{a}kT\right)\Bigg\}\Bigg], \\
F_n^{(+)}(t + x/v) &= \frac{-1}{2a^2}\sum_{k=0}^{n-1}\Bigg[(a-1)\Bigg\{E_{k+1}^{(+)}\left(-\frac{t + x/v}{a} + \frac{a+1}{a}(kT + T_1)\right) \\
&\quad - E_{k+1}^{(+)}\left(-\frac{t + x/v}{a} + \frac{a+1}{a}kT\right)\Bigg\} \\
&\quad - (a+1)\Bigg\{E_{k+1}^{(-)}\left(\frac{t + x/v}{a} + \frac{a-1}{a}(kT + T_1)\right) \\
&\quad - E_{k+1}^{(-)}\left(\frac{t + x/v}{a} + \frac{a-1}{a}kT\right)\Bigg\}\Bigg].
\end{aligned} \tag{14}$$

To show the dependence of the signal transformation on the number of cycles in the modulation packet we have considered the following model. The primary field in the form of the Gaussian pulse $F_0 = \exp(-(t - t_0 - x/v)^2/4\eta^2)$ is transformed by clearing up of the medium, when $a = \sqrt{\varepsilon/\varepsilon_1} = 1.5$. The shape of the transformed pulse after the packet of n modulation cycles is calculated according to the formulas Equations (11)–(14). In the calculation the normalized variables $t \to t/\eta$ and $x \to x/(v_2\eta)$, and the normalized duration of the medium modulation cycle $T \to T/\eta$, and the normalized duration of the disturbance interval of the cycle (on–off time ratio) $T_1 \to T_1/\eta$ are used.

The transformation of the pulse yields greater amplitudes even after the first cycle of the modulation, as it can be seen from Fig. 5 where the time behaviour of the forward and backward signals at the point where the change of the medium begins simultaneously with the signal arrival is shown. The opposite picture is observed on the inactivity interval, Fig. 6, on which

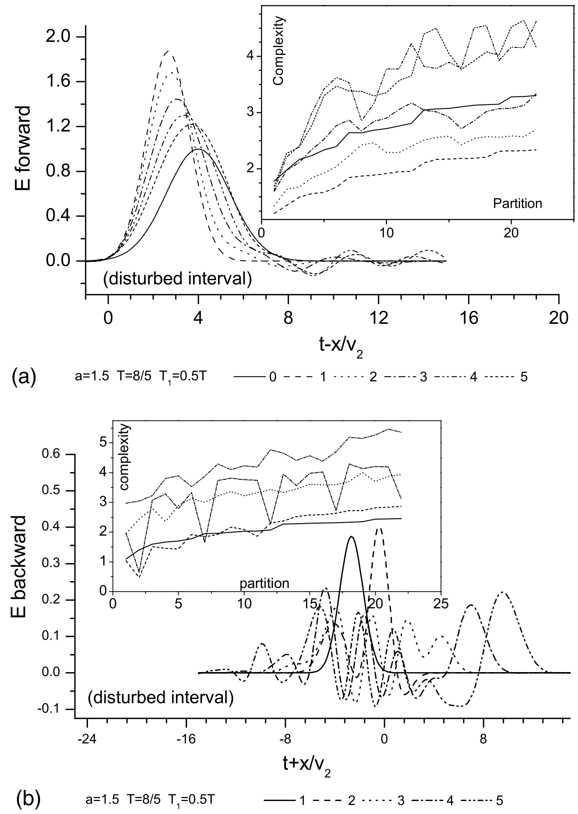


Fig. 5. Transformation of the pulse on the disturbance interval when the medium change beginning coincides with the beginning of the pulse.

darkening of the medium occurs. In both intervals the maximum of the transformed forward signal delays with each next cycle.

More complex transformation of the signal pulse is observed at the point where the medium change begins at the moment of the signal maximum that are shown in Figs. 7–9 for various values of the cycle duration.

7. Complexity of the signals

In order to estimate how complex the signals are we calculated the 'finite statistical complexity' measure of the signals. This approach of estimating the complexity of dynamical process rests on such well-known theories as Kolmogorov–Chaitin algorithmic complexity (Li and Vitanyi 1993) and Shannon (1948) entropy. The formalism is called 'computational mechanics' and was originated in the works by Crutchfield (1994), Crutchfield *et al.* (1989, 1990).

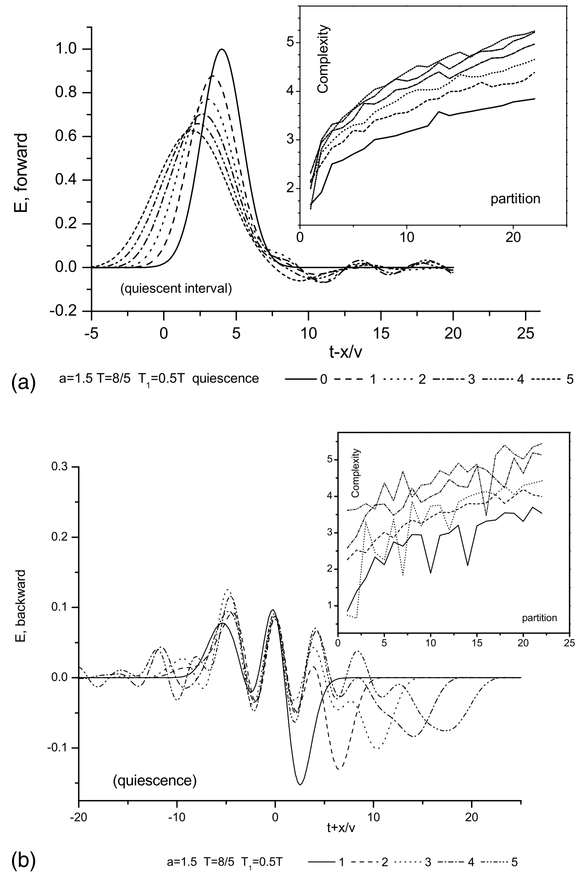


Fig. 6. Transformation of the pulse on the inactivity interval when the medium change beginning coincides with the beginning of the pulse.

First, a continuum signal is converted into a sequence of symbols from predefined alphabet. Then, the ‘symbolic dynamics’ is analysed such that the number of dynamical ‘patterns’ is extracted. The diversity of these patterns and their interrelations define the resulting complexity. In other words, this measure of complexity shows how much information is stored in the signal. It also indicates how much information is needed to predict the next value of the signal if we know all the values up to some moment in time. Intuitively, two limiting cases have zero complexity in this framework: if a signal has constant value at all times or when the signal is completely random. In both cases no information about the previous evolution needed to predict the signal at later times, it is either constant or random regardless of the previous values. All intermediate cases have a finite, non-zero value of complexity.

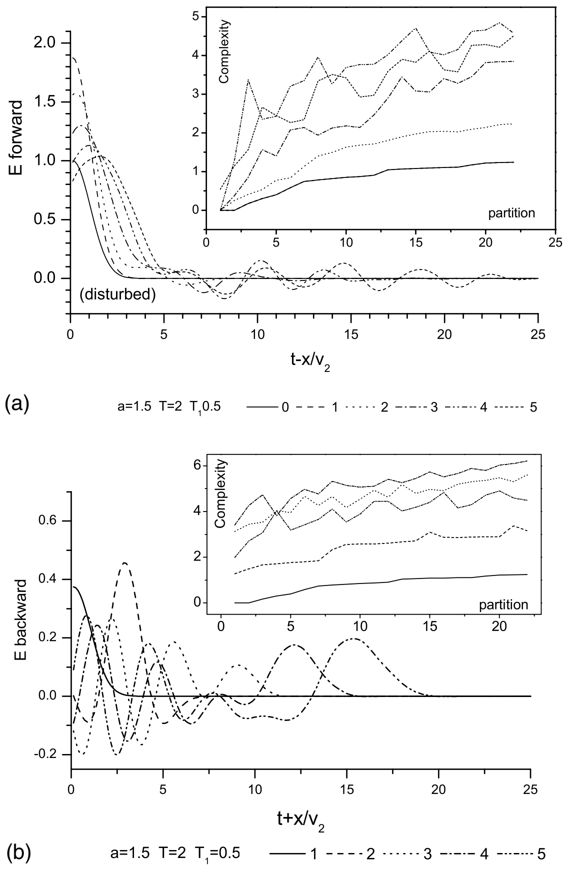
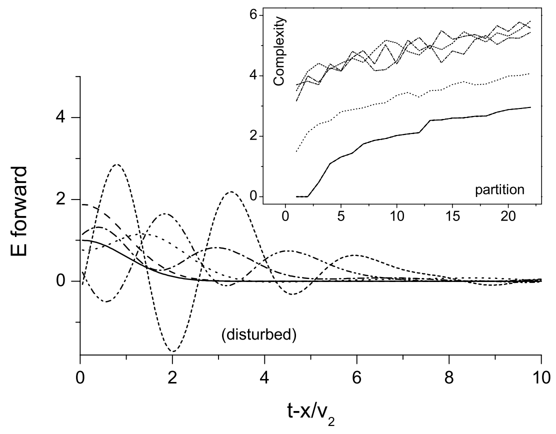


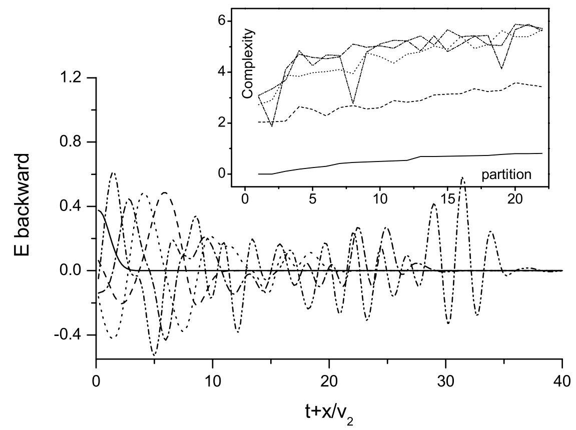
Fig. 7. Transformation of the pulse on the disturbance interval when the medium change beginning coincides with the pulse maximum – the case of the short modulation cycle.

The procedure of ‘symbolization’ is illustrated in Fig. 10. The signal is considered as a continuous function represented by the dots at the discrete experimental data points. The alphabet is constructed by the partitioning and for example for $k = 3$ consists of three symbols: $\{s_0, s_1, s_2\}$. The resulting symbolic sequence is shown at the bottom row in Fig. 10. The procedure of symbolization is described in detail in the Appendix B.

The algorithm of computing the finite statistical complexity follows the method described in Perry and Binder (1999). It consists of considering the symbolic subsequences of a finite length and analysing the ‘past’ and ‘future’ parts of them. The probabilities of various ‘futures’ are calculated as the occurrence frequencies. Different subsequences form the dynamical ‘states’ of the system and the time evolution is described as transitions between these states with some probabilities. The finite statistical complexity is calculated by the formula:



(a) a=1.5 T=4 T_1=0.5T —— 0 - - - 1 ····· 2 –·–·– 3 –··–··– 4 -·-·-·- 5



(b) a=1.5 T=4 T_1=0.5T —— 1 - - - 2 ····· 3 –·–·– 4 -·-·-·- 5

Fig. 8. Transformation of the pulse on the disturbance interval when the medium change beginning coincides with the pulse maximum – the case of the middle modulation cycle.

$$C = -\sum_i P_i \log_2 P_i, \tag{15}$$

where P_i is a probability of each dynamical state.

Dependence of calculated finite statistical complexities of the signals on the number of the modulation cycles is shown in the insets in Figs. 5–9 as plots of complexity *versus* partitions (see the algorithm of symbolization). Complexity of the forward signal in the disturbance intervals of the modulation cycle that begins at the moment of the signal arrival is shown in Fig. 5. Complexity becomes smaller after the first two modulation cycles because the signal becomes narrower. Then the signal widens and the 'tails' appear that

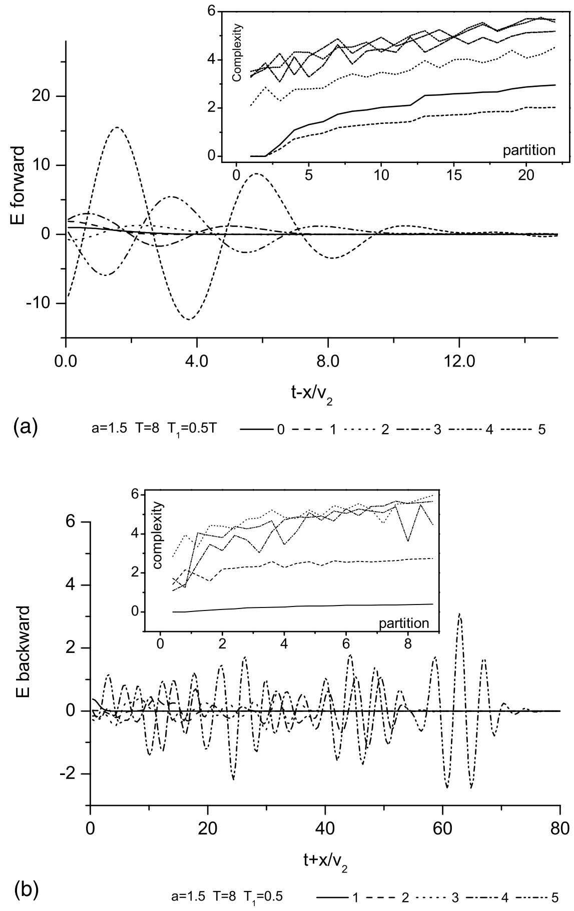


Fig. 9. Transformation of the pulse on the disturbance interval when the medium change beginning coincides with the pulse maximum – the case of the long modulation cycle.

increases signal complexity. Complexity of the forward signal in the inactivity interval changes insignificantly with number of cycles, Fig. 6. Complexity of the forward signal in the disturbance intervals of the modulation cycles that begin at the moment of the pulse maximum has a sharp jump after the first two cycles and then it stabilizes, Figs. 7–9.

Complexity of the backward signal increases after the first modulation cycle in all cases.

Low number of partitions corresponds to coarse-grained symbolizations while high values extract more information from the continuous signal. The complexity increases with the number of partitions and asymptotical behaviour at infinite number of partitions has a binary logarithm character (Cover and Thomas 1991). Therefore we analyse the relative values of the

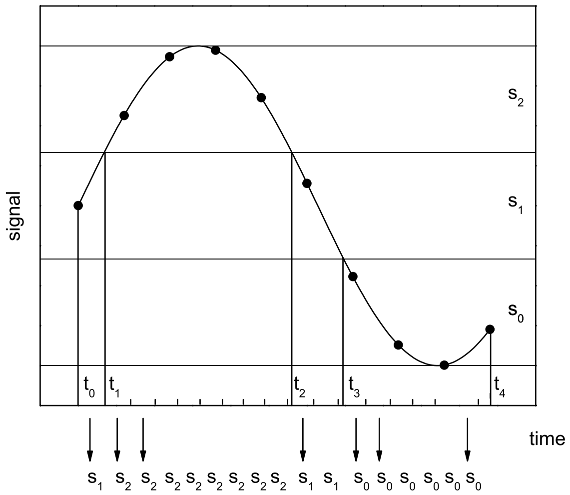


Fig. 10. Symbolization of a continuous signal. Only discrete data points (the dots) representing the continuous signal (the solid curve) are available.

complexity of different signals at some specified high value of the number of partitions. The values of complexity for 25 partitions are reported in Table 1.

Overall the complexity of the signals increases with the number of cycles. However, in those cases when the signals get narrower keeping their shape

Table 1.

Number of cycles	0	1	2	3	4	5
$T = 8/5$						
Disturbance interval						
Forward	3.30	2.33	2.70	3.34	4.30	4.60
Backward	–	2.45	2.87	3.93	4.20	5.40
Inactivity interval						
Forward	3.82	4.38	4.65	4.96	5.21	5.23
Backward	–	3.53	3.99	4.42	5.13	5.44
$T = 2$						
Disturbance interval						
Forward	1.24	1.24	2.23	3.84	4.57	4.51
Backward	–	1.24	3.16	5.60	6.20	4.40
$T = 4$						
Disturbance interval						
Forward	2.95	2.95	4.07	5.57	5.44	5.81
Backward	–	0.81	3.42	5.66	5.61	5.72
$T = 8$						
Disturbance interval						
Forward	2.95	2.02	4.51	5.19	5.67	5.55
Backward	–	0.40	2.74	5.97	5.65	4.47

the complexity may decrease (e.g. after the first cycle for the case of $T = 8$ and 8/5). At higher number of cycles the increase of the complexity is explained by the appearance of the complicated tails at longer times even when the main signal body remains narrow. The backward signals can be either more complex than the forward ones or less complex as in the inactivity interval for $T = 8/5$ or in the disturbance interval for $T = 4$.

In many cases the calculated complexity allows to quantitatively estimate the relative informational contents of the signals. This is especially true in the situations when it is impossible to judge by eye either when the signals have simple and very similar shape, as in Figs. 5 and 6, or they are too complex to compare, as in Figs. 7–9.

8. Conclusion

The Volterra integral equation approach allows obtaining an exact solution to a problem of an accurate modelling of a transformation of an electromagnetic signal caused by a finite sequence of time cycles of medium parameters change. Each modulation cycle has a form of a rectangular pulse and consists of disturbance and inactivity intervals. Investigations show significant transformation of the electromagnetic signal by only a short sequence of cycles. It is shown that the forward signals have greater amplitude in the intervals of medium clearing up. It is also shown an appearance of the backward (reflected) signals that are inevitable result of the medium time change.

The calculated measure of the complexity of the transformed electromagnetic signal increases with the number of modulation cycles. This measure allows to quantitatively estimate the relative informational contents of the signals. This is especially true in the situations when it is impossible to judge by eye either when the signals have simple and very similar shape or they are too complex to compare.

Appendix A

An action of the resolvent operator is determined by the integral

$$RF_{n-1} = \int_{(n-1)T}^{t} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x' \langle t,x|R|t',x'\rangle F_{n-1}(t',x'). \tag{16}$$

The kernel of this operator is given by the inverse Laplace transform (Nerukh *et al.* 1998)

$$\langle t, x|R|t', x'\rangle = -\frac{1}{2v_1}\int\limits_{\eta-i\infty}^{\eta+i\infty}\frac{dp}{2\pi i}\frac{(p-b)[(p+b)-a^2(p-b)]}{\sqrt{p^2-b^2}} \times \exp\left((p-b)(t-t') - \frac{|x-x'|}{v_1}\sqrt{p^2-b^2}\right) \quad (17)$$

where $\eta > b$ and the square root is taken such that $\mathrm{Re}(p^2 - b^2)^{1/2} > 0$.

In the case when the conductivity of the medium is equal to zero the kernel of the resolvent has a simple form

$$\langle t, x|R|t', x'\rangle = -\frac{1-a^2}{2}\frac{\partial}{\partial t}\delta(v_1(t-t') - |x-x'|) \quad (18)$$

Appendix B

The algorithm for the symbolization is as follows:

(1) Find the intersection points of a signal with the partition lines (for example, t_1, t_2, t_3 in Fig. 10), takes the first and the last points of the signal as well (t_0, t_4).
(2) Find the smallest interval in time Δt (equals to the length of $[t_0, t_1]$ for this example).
(3) For each interval find the number of symbols produced by this interval by dividing its length by Δt.
(4) Form the final sequence by choosing the symbol from the partition space where the signal falls between the intersection points (Fig. 10).

It is important to consider the intersection points t_i as time interval boundaries and not the data points themselves. Otherwise, if the data points do not fall in the points of natural periodicity an artificial randomness is introduced into the final symbolic sequence. Also, generating a sequence of repeating symbols like the ones on the $[t_1, t_2]$ interval to our belief preserves more information from the original signal.

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