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Wave chaotic behaviour generated by linear systems

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Abstract It is shown that regimes with dynamical chaos are inherent not only to nonlinear system but they can be generated by initially linear systems and the requirements for chaotic dynamics and characteristics need further elaboration. Three simplest physical models are considered as examples. In the first, dynamic chaos in the interaction of three linear oscillators is investigated. Analogous process is shown in the second model of electromagnetic wave scattering in a double periodical inhomogeneous medium occupying half-space. The third model is a linear parametric problem for the electromagnetic field in homogeneous dielectric medium which permittivity is modulated in time.

Keywords Electromagnetic linear problems · Transients · Chaos

1 Introduction

It is generally accepted that chaotic dynamics and chaotic characteristics are inherent to nonlinear systems only. Moreover, the emergence of dynamic chaos requires the appearance of a local instability in the system and this instability has to be nonlinear (Lichtenberg and Lieberman 1983). In this paper we show that the latter requirement needs further elaboration.

The dynamics of a system in quantum and classical regimes is a good example of the chaos (in classical description) resulting from initially linear equations of quantum mechanics. Non-trivial transformations of the latter obscure the nonlinearity of the classical equations of motion. The investigations directed for establishing the correspondence between the quantum and classical descriptions of chaotic systems are active for long time and they even have

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a special name, "quantum chaos". Numerous publications are devoted to quantum chaos (see, for example, [H.-J. Stoeckmann, Quantum Chaos: An Introduction, Cambridge University Press, 1999]; [M.C. Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer, 1990]). The purpose of this work is to turn reader's attention to the fact that such situations are frequent and require careful treatment.

Indeed, the equations of geometrical optics, which describe the chaotic dynamics of light beams, are also derived from the linear Maxwell's equations. The change of the dependent variables can transform the initially linear equations into nonlinear ones that can have chaotic solutions and, therefore, should be investigated accordingly. This fact seems to escape researches' attention. Nevertheless, it should be taken into account in the analysis of a wide range of physical processes.

In this paper we consider the simplest linear physical models where the described peculiarity in the dynamics can be realized and we show that some physical variables characterising such systems can exhibit chaotic dynamics (see also Buts 2006). Importantly, these types of models are often used in the description of various physical processes.

First, we consider three linear interacting oscillators. Under some approximations the same equations can be used to model electromagnetic wave scattering by double periodical inhomogeneous medium occupying a half-space. The first period of inhomogeneity corresponds to such a reciprocal lattice vector that the incident wave scattered by this period of inhomogeneity generates a wave of the minus first order diffraction. Conditions for the excitation of the same wave diffraction by the second period of the inhomogeneity are realized with some detuning. Thus, we have a case of three-wave dynamical diffraction. Equations describing the dynamics of the complex amplitudes of these three interacting waves are linear. Therefore, their dynamics and all characteristics are, in general, regular. However, in the case of a weak coupling between the waves a system of reduced equations for the real amplitudes and phases can be obtained. This reduced system is nonlinear and it is simpler than the initial one. This allows to obtain analytical criteria for the onset of the dynamical chaos. This system has been investigated analytically and numerically in the present paper. Qualitative agreement between the analytical and numerical results is obtained, namely, a stochastic instability develops as soon as the conditions for the intersections of the heteroclinic trajectories are fulfilled. In this case the spectra broaden, the correlation function decreases and the real part of the maximal Lyapunov exponent becomes positive.

Second, we show that some important characteristics in the solutions of pure linear problems can have chaotic dynamics. It is caused by the fact that these characteristics themselves are nonlinear. As an example, a linear parametric problem of the electromagnetic field in the homogeneous dielectric medium which permittivity is modulated in time beginning from some moment is considered. If the initial field is a plane monochromatic wave and the modulation is in the form of rectangular pulses then the problem has an exact analytical solution which describes the temporal process of the wave transformation. Since, under the modulation, the permittivity changes abruptly from the initial value to the new one and back then the transformation process consists of a progressive repetition of the known effect of each wave splitting onto a pair of forward and backward propagating waves at each jump of the permittivity. Relationships between these wave amplitudes are obtained exactly in analytical form. They reveal a controlling sequence which determines the whole behaviour of the relations with time. The analysis shows that the temporal course of this sequence can have distinctly non-regular behaviour, which chaotic character is confirmed by the Lamerey diagram, the calculation of the Hurst exponent, the signal complexity, and the Lyapunov exponent. It is shown that the Hurst exponent takes the values corresponding to the white noise, the signal complexity rises and the Lyapunov exponent becomes positive.

2 Dynamic chaos in the interaction of three linear oscillators

Here we consider the simplest but very important physical system of three coupled linear oscillators, in which the regime of chaotic dynamics can exist. The system Hamiltonian has the form:

$$H = \sum_{i=0}^{2} \left[p_i^2/2 + \omega_i^2 q_i^2/2 \right] + q_0 \sum_{i=1}^{2} \mu_i q_i \tag{1}$$

We consider the case when two oscillators are identical but the frequency of the third one differs slightly from the frequency of the other two: $\omega_1 = \omega_0 = \omega$, $\omega_2 = \omega + \Delta\omega$. We also assume that the interaction coefficients are small. In this case the dynamics is described by the following system of equations:

$$\ddot{q}_0 + q_0 = -\mu_1 q_1 - \mu_2 q_2,\ \ddot{q}_1 + q_1 = -\mu_1 q_0,\ \ddot{q}_2 + (1+\delta)\, q_2 = -\mu_2 q_0, \tag{2}$$

where $\dot{q} \equiv dq/d\tau, \tau = \omega t, \delta \equiv 2\Delta\omega/\omega,\ \mu_i \equiv \mu_i/\omega^2,\ \mu_i \ll 1, \delta \ll 1$, and the terms proportional to $(\Delta\omega/\omega)^2$ are neglected. The dimensionless coefficients of the interaction are introduced in (2).

Taking into account the fact that the right sides (the factors of connection) in the Eq. 2 are small the solutions can be sought in the form:

$$q_i = A_i(\tau) \exp\,(i\omega_i t), \tag{3}$$

where the dependence of the complex amplitudes on time is caused by the connection between the oscillators. If this connection is small then the amplitudes are slow varying functions and the averaging method can be used. As a result we obtain the following system of reduced equations for the amplitudes:

$$\begin{aligned} 2i\dot{A}_0 &= -\mu_1 A_1 - \mu_2 A_2 \exp(i\delta\tau) \\ 2i\dot{A}_1 &= -\mu_1 A_0 \\ 2i\dot{A}_2 &= -\mu_2 A_0 \exp(-i\delta\tau). \end{aligned} \tag{4}$$

The connection between the complex amplitudes follows from (4)

$$\frac{d}{d\tau}\left[A_0^2 - A_1^2 - A_2^2\right] = 2 \cdot \mu_2 \cdot A_0 A_2 \sin\,(\delta\tau)$$

It follows from this equation that the system (4) has only one degree of freedom if detuning of the frequency is equal to zero ($\delta = 0$). Therefore, the development of dynamical chaos in this system is impossible. In other cases the detuning determines a distance between nonlinear resonances.

For further analysis we represent the complex amplitudes in the form:

$$A_i(\tau) = a_i(\tau) \exp(i\varphi(\tau)) \tag{5}$$

where a_i, φ_i are real amplitudes and real phases.

Substitution of (5) into (4) gives the following system of equations:

$$\begin{aligned}
\dot{a}_0 &= -(\mu_1/2)\, a_1 \cdot \sin(\Phi) - (\mu_2/2) \cdot a_2 \cdot \sin(\Phi_1)\,,\\
\dot{a}_1 &= (\mu_1/2)\, a_0 \cdot \sin(\Phi)\,,\\
\dot{a}_2 &= (\mu_2/2)\, a_0 \cdot \sin(\Phi_1)\\
\dot{\Phi} &= (\mu_1/2)\left(\frac{a_0}{a_1} - \frac{a_1}{a_0}\right)\cos(\Phi) - (\mu_2/2)\left(\frac{a_2}{a_0}\right)\cos(\Phi_1)\,,\\
\dot{\Phi}_1 &= (\mu_2/2)\left(\frac{a_0}{a_2} - \frac{a_2}{a_0}\right)\cos(\Phi_1) - (\mu_1/2)\left(\frac{a_1}{a_0}\right)\cos(\Phi) + \delta,
\end{aligned} \tag{6}$$

where $\Phi \equiv \varphi_1 - \varphi_0$, $\Phi_1 \equiv \varphi_2 - \varphi_0 + \delta\tau$.

The system (6) is a simplified one compared to the initial system (2) but it is nonlinear and the dynamics of this system can be chaotic.

2.1 Analytical criterion for the onset of dynamical chaos

The analytical conditions for the dynamics of the nonlinear system (6) to be chaotic are of great practical interest. To find these conditions we initially assume that there are only two oscillators, the first and the second, and the third one is absent ($A_2 = 0$). In this case it follows from (4) that

$$A_0^2 - A_1^2 = \text{const}, \tag{7}$$

therefore, the dynamics of the complex amplitudes A_0 and A_1 is very simple. They oscillate with the frequency $\mu/2$. On the contrary, the dynamics of the real amplitudes a_i and the real phases Φ and Φ_1 is more complex. Indeed, the equation for the phase Φ is nonlinear

$$\ddot{\Phi} = -\frac{\mu_1^2}{8}\left[\frac{\left(a_0^2 + a_1^2\right)^2 + \left(a_0^2 - a_1^2\right)^2}{a_0^2 a_1^2}\right]\sin(2\Phi) \tag{8}$$

and represents a well known equation for the mathematical pendulum. Thus, the behavoiur of the real amplitudes a_i is known qualitatively and consists of the following. Let only the first oscillator have non-zero energy at the initial moment of time. Then the oscillation amplitude of the second one is equal to zero. After some time interval ($\tau \sim 2/\mu_1$) the first oscillator amplitude will become zero, while the second oscillator amplitude will achieve the maximum value which is equal to 1. Therefore, the absolute value of the expression in the square brackets in (8) is always greater than 1. Consequently, the minimal width of the nonlinear resonance can be estimated as $\Delta \sim \mu_1$.

Let us consider now the situation when the second oscillator is absent ($A_1 = 0$) and there is an interaction of the first and the third oscillators. In this case the equation for the phase looks like

$$\ddot{\Phi}_1 = -\frac{\mu_2^2}{8}\left[\frac{\left(a_0^2 + a_2^2\right)^2 + \left(a_0^2 - a_2^2\right)^2}{a_0^2 a_2^2}\right]\sin(2\Phi_1) - \frac{\mu_2\delta}{2}\left(\frac{a_0^2 - a_2^2}{a_0 a_2}\right)\sin(\Phi_1). \tag{9}$$

The Eq. 9 differs from the Eq. 8 by the presence of the last term which is caused by detuning δ between the frequencies. The Eq. 9 represents the equation of nonlinear oscillator also but its structure is considerably more complex than the structure of the nonlinear Eq. 8. However, the qualitative dynamics of the real amplitudes a_i are also known. Therefore, as in the previous case, we can estimate the minimal width of the nonlinear resonance as $\Delta_1 \sim \mu_2$.

It is natural to expect that when the nonlinear resonances are overlapped, i.e. when the condition $(\mu_1 + \mu_2) > \delta$ is fulfilled, the dynamics of the system (6) will be chaotic. In the next section we show that the numerical investigations confirm this conclusion.

2.2 Numerical investigations

The systems of Eqs. 2, 4 and the system (6) were investigated numerically. In all cases the dynamics of the system described by the original systems of Eqs. 2 and 4 is regular: the spectrum is narrow enough; the correlation functions oscillate, but their amplitudes do not decrease; the maximum Lyapunov index is very small ($|\lambda| < 10^{-4}$).

The dynamics of the system (6) is, however, qualitatively different. If the conditions for overlapping the nonlinear resonances are fulfilled then the dynamics becomes chaotic. Figures 1–5 represent the numerical results obtained for the system parameters $\delta = 0.017$, $\mu_1 = 0.02$, $\mu_2 = 0.012$, which provide the fulfilment of the conditions for the onset of the dynamical chaos. The time behaviour of the real amplitude a_0 in Fig. 1 and the phase Φ in Fig. 2 show significant jumps of the phase caused by passing of the real amplitudes through their minimal values.

The spectrum and the autocorrelation function for the first oscillator amplitude are shown in Figs. 3 and 4. The spectrum is significantly broader in comparison with the case of the original system. The correlation function decreases during the whole time interval of

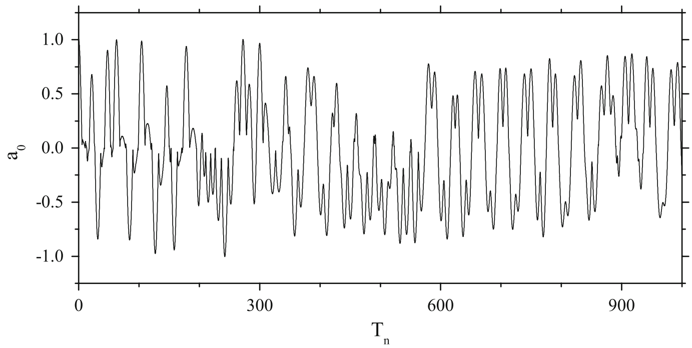

Fig. 1 The evolution of the real amplitude of the first oscillator at $\delta = 0.1$, $\mu_1 = 0.2$, $\mu_2 = 0.2$. The conditions for the chaos onset are fulfilled

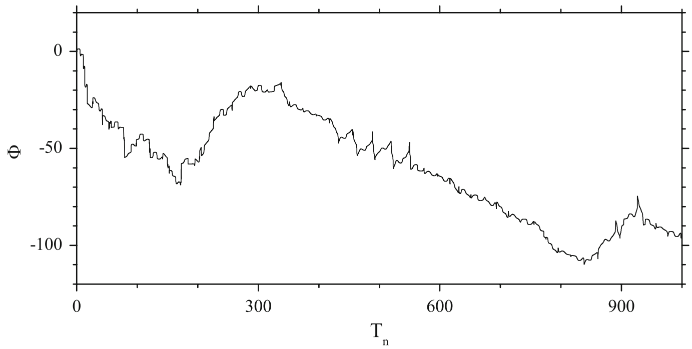

Fig. 2 The dynamics of the phase at $\delta = 0.1$, $\mu_1 = 0.2$, $\mu_2 = 0.2$

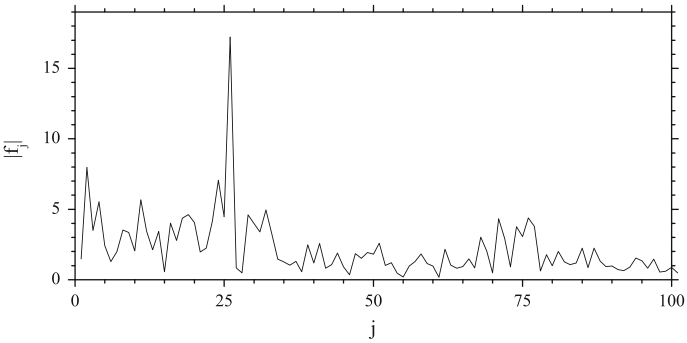

Fig. 3 The spectrum of the first oscillator at $\delta = 0.1$, $\mu_1 = 0.2$, $\mu_2 = 0.2$

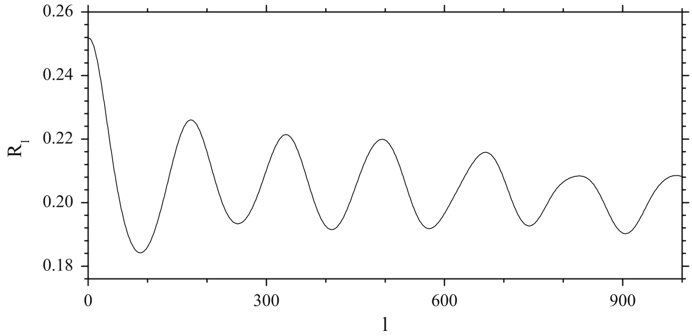

Fig. 4 The autocorrelation function at $\delta = 0.1$, $\mu_1 = 0.2$, $\mu_2 = 0.2$

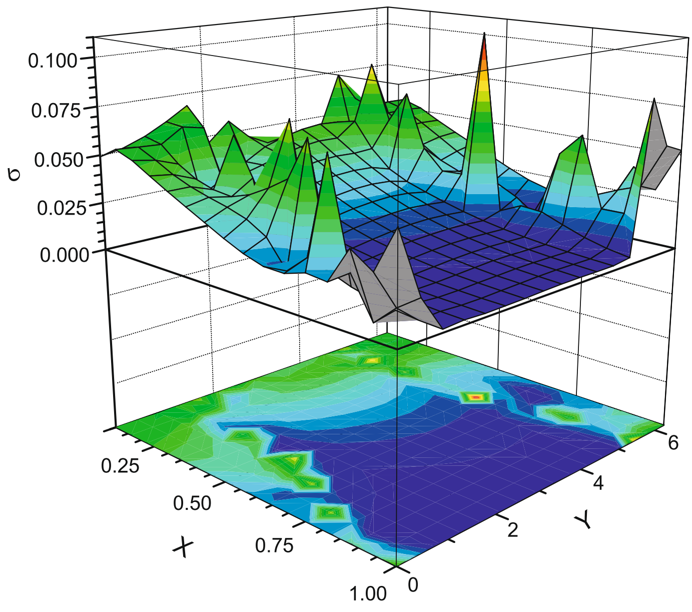

Fig. 5 The main Lyapunov index as function of the location initial points in phase space at $\mu_1 = \mu_2 = 0.2$, $\delta = 0.1$

observations. Thus, the dynamical chaos regime is realized for these values of the system parameters. It is confirmed by the maximum Lyapunov index. In Fig. 5 this index is plotted as a function of the initial points locations on the (Φ, a_0) plane. The Lyapunov index is positive in the whole area considered.

3 Waves scattering in the medium with weak periodic inhomogeneous

Properties similar considered case of regular or chaotic dynamics in the system of three linear coupled oscillators take place in a large number of other linear systems. Here we consider a model of an electromagnetic wave scattering on a non-uniform dielectric medium filling the half-space ($z > 0$ for unambiguity). The wave comes from the upper ($z < 0$) homogeneous half-space. The wave electrical field satisfies the well known wave equation

$$\Delta E + k^2 \varepsilon E = 0, \tag{10}$$

where $k = \omega/c$.

We consider the case when the permittivity is described by the formula

$$\varepsilon = 1 + \sum_{i=1}^{2} \mu_i \cdot \cos(\vec{\kappa}_i \cdot \vec{r}) \,, \tag{11}$$

It is assumed that the heterogeneity is small, $\mu_i \ll 1$. In this case the electromagnetic wave scattering gives diffracted waves of the minus first order of diffraction. The complete field can be represented in the form

$$E = \sum_{i=0}^{2} E_i = \sum_{i=0}^{2} A_i(z) \cdot \exp\left(i\vec{k}_i\vec{r}\right). \tag{12}$$

The first term in (12) corresponds to the falling wave, the second and third ones correspond to the minus first order of diffraction by the first and the second heterogeneities accordingly. Let the relation between the wave vectors satisfy the following expressions

$$\vec{k}_1 = \vec{k}_0 - \vec{\kappa}_1, \vec{k}_2 = \vec{k}_0 - \vec{\kappa}_2 + \vec{\delta}, \vec{\delta} = (0, 0, \delta) \,. \tag{13}$$

In this model we consider the case when all the wave amplitudes depend only on the coordinate z directed into the lower half-space. Substituting (11) and (12) into the wave equation 10 and applying averaging to find the slow varying amplitudes of the interactive waves we obtain the following system of equations:

$$\begin{aligned} 2iA_0' &= \mu_1 A_1 + \mu_2 A_2 \exp(i\delta z), \\ 2iA_1' &= \mu_1 A_0 \cdot (k_{0z}/k_{1z}) \,, \\ 2iA_2' &= \mu_2 A_0 \cdot (k_{0z}/k_{2z}) \exp(-i\delta z), \end{aligned} \tag{14}$$

where $A' \equiv dA/dz$ and the following dimensionless parameters and independent variables are introduced: $\mu_1 \equiv \mu_1 \cdot k/k_{0z}$, $\mu_2 \equiv \mu_2 \cdot k/k_{0z}$, $\delta \equiv \delta/k$, $z \equiv k \cdot z$.

The system (14) is similar to the system of Eq. 4 if the derivative with respect to time is substituted by the derivative with respect to the coordinate z. Therefore, the dynamics of the systems (4) and (14) has the same qualitative description. They both have the areas of parameters, in which their behaviour is chaotic. The chaos criterion, for example for the system (14), in the symmetric case ($\mu_1 \approx \mu_2 = \mu$) is given by the inequality:

$$\delta < (k^2 \cdot \mu)/4\sqrt{k_{1z} \cdot k_{2z}}. \tag{15}$$

4 Parametric phenomenon

As the third example, a linear parametric problem for the electromagnetic field in the homogeneous dielectric medium which permittivity is modulated in time beginning from some moment is investigated. Parametric phenomena in active media have been attracting much attention for a long time in connection with the transformation of electromagnetic waves by the time variation of the medium parameters. In the systems with distributed parameters main features of the wave transformation by the medium nonstationarity can be revealed when a simple law changes the medium parameters such that an exact solution of the problem can be constructed.

In this paper the electromagnetic wave transformation in a medium which parameters undergo changes in a form of a finite packet of periodic rectangular pulses is considered. Regularity of the transformation is estimated by three characteristics, the Hurst's index (Hurst et al. 1965), the signal complexity (Crutchfield and Young 1989) and the Lyapunov exponent (Lichtenberg and Lieberman 1983; Kuznetsov 2000).

Such a model allows to carry out the exact investigation of the process. If the initial field is a plane monochromatic wave and the modulation is in the form of rectangular pulses then the problem has an exact analytical solution which describes the temporal process of the wave transformation. Since under the modulation the permittivity changes abruptly from the initial value to the new one and back then the transformation process consists of a progressive repetition of the known effect of each wave splitting onto a pair waves at each jump of the permittivity. These waves propagate in the opposite directions and their frequencies change by a jump periodically also from the initial value to the new one and back. The wave amplitudes undergo sophisticated transformations with the increase of time (the number of the permittivity jumps). However, they are the solutions to the linear electrodynamics wave equations with variable coefficients. Since the essential feature in this process is the wave splitting (the wave reflection in time) then it is interesting to observe a change of relationship between the amplitudes of the forward and backward waves in the transformation process. These relations are obtained exactly in an analytical form.

4.1 Wave transformation under medium modulation

4.1.1 Step by step wave transformation

We consider an unbounded dielectric dissipative medium, the permittivity and conductivity of which are modulated according to the law of a finite packet of N rectangular periodic pulses:

$$\varepsilon(t) = \varepsilon + (\varepsilon_1 - \varepsilon)\sum_{k=1}^{N}\{\theta(t-(k-1)T)\ -\theta(t-T_1-(k-1)T)\}$$

$$\sigma(t) = \sigma_1\sum_{k=1}^{N}\{\theta(t-(k-1)T)\ -\theta(t-T_1-(k-1)T)\} \tag{16}$$

Here, $\theta(t)$ is the Heaviside unit function, T is the duration of the period of the parameters change, T_1 is the duration of the disturbance interval, in which the medium permittivity and conductivity receive new magnitudes ε_1 and σ_1. Further, we normalize all time variables to a frequency ω of the initial wave, $t \to \omega t$. This wave exists before zero moment of time, the moment when the modulation commences, and is given by the function

$E_0(t, x) = \exp[i(t - kx)]$ in normalized variables. Each time jump of the medium properties changes the electromagnetic field, such that it is described by the functions E_n on the disturbance intervals and by F_n on the inactivity intervals where the medium permittivity and conductivity return to the initial magnitudes.

After beginning the modulation by the disturbance interval the initial wave is splitting into two, forward and backward, waves $E_1 = \exp(-st - ikx)\left[C_1 \exp(iqt) + D_1 \exp(-iqt)\right]$ with new amplitudes and the new normalized frequency $q = (a^2 - s^2)^{\frac{1}{2}}$ where $a^2 = \varepsilon/\varepsilon_1$, $s = \sigma_1/\omega\varepsilon_0\varepsilon$, and ε_0 is the vacuum permittivity. On the remaining undisturbed interval of this first modulation period the field splitting into two waves remains $F_1 = \exp(-ik)\left[A_1 \exp(it) + B_1 \exp(-it)\right]$ but the frequency returns to the original one.

The field on the other disturbance intervals consists also of two, forward and backward, waves $E_n = \exp(-st - ikx)\left[C_n \exp(iqt) + D_n \exp(-iqt)\right]$ of changed frequency while the field on the inactivity intervals consists of two waves also $F_n = \exp(-ikx)\left[A_n \exp(it) + B_n \exp(-it)\right]$ but of the unchanged frequency. Therefore, the transformed field at any moment t in the Nth modulation period is given by the formula

$$E(t, x) = \sum_{n}^{N} E_n\theta(t - (n-1)T) - \sum_{n}^{N} \{E_n\theta(t - nT_1 - (n-1)(T - T_1)) - F_n\theta(t - nT_1 - (n-1)(T - T_1)) + F_n\theta(t - nT)\} \tag{17}$$

The exact expressions for the direct and the inverse secondary wave amplitudes are given in (Nerukh 1999; Ruzhytska et al. 2003)

4.1.2 Parameters of transformation

Wave reflection in time can be characterized by a temporal reflectance as the ratio of the backward (inverse) and forward (direct) wave amplitudes. In (Nerukh 1999; Ruzhytska et al. 2003) it is shown that these ratios are determined by the expressions:
on the disturbance intervals

$$w_N = \frac{D_N}{C_N} e^{-i2(N-1)qT} = \frac{\{p_2 + (p_1 - p_2)r_{N-1}\}\alpha_{21} + p_1 p_2 \alpha_{22}}{\{p_2 + (p_1 - p_2)r_{N-1}\}\varepsilon_{11} + p_1 p_2 \alpha_{12}}; \tag{18}$$

on the inactivity intervals

$$p_N = \frac{B_N}{A_N} e^{-i2NT} = \frac{p_1 p_2}{p_2 + (p_1 - p_2)r_N}, \quad N \geq 2. \tag{19}$$

Here, $p_1 = -h/m$, $p_2 = -h(m + m^*)/(hh^* + m^2)$, $A_1 = m\exp(-iT)$, $B_1 = -h\exp(iT)$, $\alpha_{11} = q + 1 + is$, $\alpha_{12} = q - 1 + is$, $\alpha_{21} = q - 1 - is$, $\alpha_{22} = q + 1 - is$, and

$$m = \frac{1}{2q}\left[2q\cos(qT_1) + i(a^2 + 1)\sin(qT_1)\right]\exp[-sT_1 + i(T - T_1)] \tag{20}$$

$$h = i\frac{1}{2q}\left(a^2 - 1 - i2s\right)\sin(qT_1)\exp[-sT_1 - i(T - T_1)]. \tag{21}$$

As it follows from (18) and (19) the behaviour of the ratios w_N and p_N is governed by the sequence

$$r_{N+1} = 4u^2/(4u^2 - r_N), \tag{22}$$

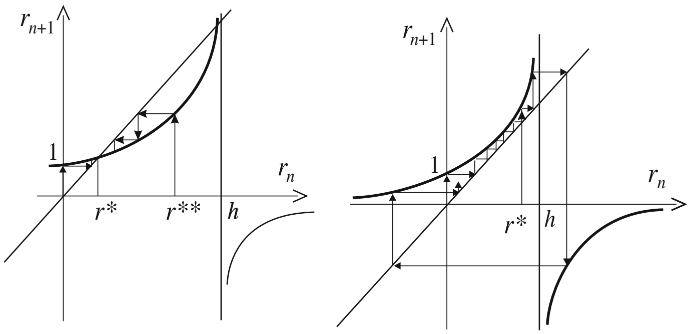

Fig. 6 The Lamerey diagram for the controlling sequence: (left) for the generalized parameter $u > 1$; (right) for $u < 1$

which is controlled by the generalized parameter

$$u = \cos(qT_1)\cos(T - T_1) - \frac{a^2 + 1}{2q}\sin(qT_1)\sin(T - T_1). \tag{23}$$

The analysis shows that the temporal course of the controlling sequence can have a monotone, non-monotone but regular, and distinctly non-regular character depending on the generalized parameter value. In the last case the non-regularity has a form of chaotic behaviour that is visually confirmed by the Lamerey diagram, Fig. 6. There are long intervals in the sequence of the modulation periods where r_N changes almost regularly. After this interval the relatively short intervals of strong irregular behaviour of r_N take place. Larger deviations of u^2 from 1 lead to more irregular behaviour of r_N. This phenomenon can be termed "quasi-intermittency".

The similar behaviour takes place for the transformed field. If $u^2 > 1$ it has the regular character with time as well as the sequence r_N, Fig. 7a. Otherwise, if $u^2 < 1$, the sequence as well as the field have irregular behaviour, Fig. 7b.

4.2 Characteristics of chaotic dynamics

4.2.1 The Hurst's index

The presence of the quasi-intermittency can be confirmed by the Hurst's method (Hurst et al. 1965), according to which the time series of r_n is characterized by the Hurst's index H, which is determined as the asymptotic value of the function

$$H \sim \ln(R_n/S_n)/\ln n \tag{24}$$

where $R_n = \max_{1\le k\le n} X(k,n) - \min_{1\le k\le n} X(k,n)$, and

$$X(k,n) = \sum_{i=1}^{k}(r_i - \langle r\rangle_n),\ \langle r\rangle_n = \frac{1}{n}\sum_{i=1}^{n} r_i,\ S_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(r_i - \langle r\rangle_n)^2}. \tag{25}$$

For the white noise (a completely uncorrelated signal) this index equals to $H = 0.5$. The value $H > 0.5 (H < 0.5)$ is associated with the long-range correlation when the time series exhibits persistence (antipersistence).

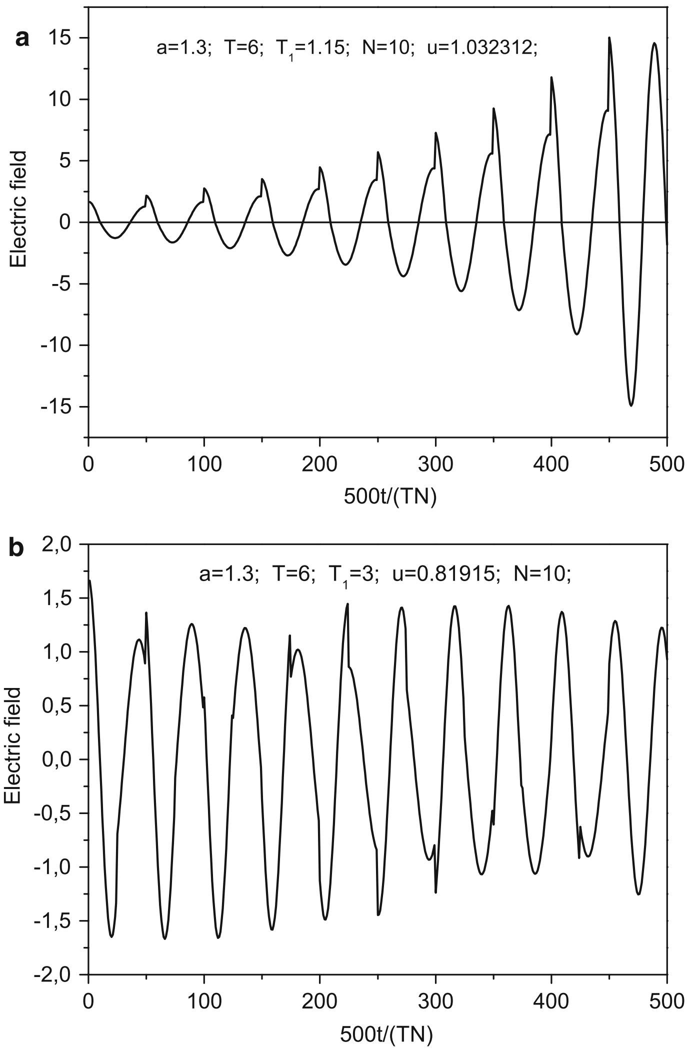

Fig. 7 The behaviour of the transformed field with time. (**a**) Parametric amplification, (**b**) irregular changing

The chaotic character of r_N is demonstrated by the calculation of the Hurst's index. It can be seen from Fig. 8 that the decrease of the generalized parameter leads to a situation when the Hurst's index takes magnitudes corresponding to the white noise, $H > 0.5$.

4.2.2 The signal complexity

The r_N behaviour can be also characterised by a complexity measure (Crutchfield and Young 1989). This measure of complexity shows how much information is stored in the signal and how much information is needed to predict the next value of the signal if we know all the values up to some moment in time. In two limiting cases, when a signal has constant value at all times and when the signal is completely random, a complexity is equal to zero in this

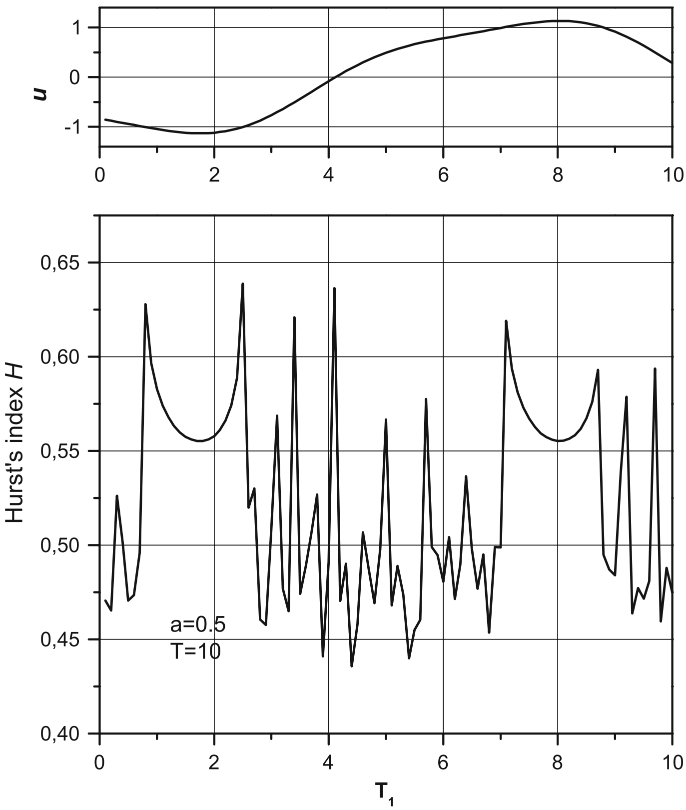

Fig. 8 The detailed behaviour of Hurst's index versus the duration of the disturbance interval (the parameter u is given in the upper diagram)

framework because of no information about the previous evolution needed to predict the signal in both cases. All intermediate cases have a finite, non-zero value of the complexity.

The algorithm of computing the finite statistical complexity (Nerukh et al. 2002) follows the method originated in the works by Crutchfield and others and it consists of considering the symbolic subsequences that form the dynamical 'states' of the system and the time evolution, which is described as transitions between these states with some probabilities P_i. The finite statistical complexity is calculated by the formula:

$$C = -\sum_i P_i \log_2 P_i. \tag{26}$$

The dependence of this measure on the modulation period shows a correlation between the complexity and the generalized parameter u, Fig. 9. The electromagnetic signal is regular and its complexity drops to zero when the absolute value of the generalized parameter u (the dash-dot sine-like line) becomes greater than 1.

In this case the value of the H index is typical for the regular behaviour. The complexity drops to zero when the Hurst's index deviates notably from the value of 0.5 (in average) that corresponds to regular behaviour of the signal, Fig. 8. Thus, the correlation exists between the Hurst's index and the complexity of the signal and it corresponds to the behaviour of these two characteristics and the sequence r_n. Both characteristics correlate with the generalized

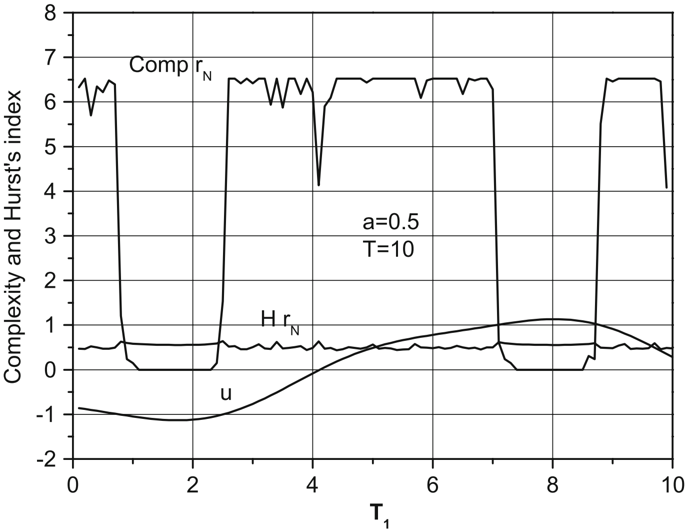

Fig. 9 The behaviour of the complexity versus the modulation periods (the parameter u is shown by the dash-dot sine-like line)

parameter u. This is true for both cases when the medium becomes more, $a < 1$, or less, $a > 1$, optically dense on the disturbance intervals.

4.2.3 The Lyapunov exponent

The behaviour of r_n can be quantified by the Lyapunov exponent. Let us derive the estimation of this exponent as in (Kuznetsov 2000). For this, we consider two nearby trajectories r_n and $r'_n = r_n + \tilde{r}_n$ of recurrence mapping $r_{n+1} = f(r_n)$ given by the formula (22). Using the Taylor's series expansion of (22) one can derive

$$\tilde{r}_{n+1} = \tilde{r}_1/(u^{2n} + B_1 u^{2(n-1)} + \cdots + B_{n-1}) \tag{27}$$

where B_i are some coefficients that do not depend on u.

The Lyapunov exponent is determined by the evolution of the small disturbance in linear approximation as

$$\Lambda = \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \|\tilde{r}_n\| \tag{28}$$

Substitution of $\tilde{r}_n$ from (27) gives, at least for $u^2 > 1$,

$$\Lambda \approx \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \left\|\tilde{r}_1/u^{2(n-1)}\right\| = \overline{\lim_{n\to\infty}} \left(\ln \|\tilde{r}_1\|/n - \frac{n-1}{n} \ln \left\|u^2\right\| \right) = -\ln \left\|u^2\right\| \tag{29}$$

Therefore, the Lyapunov exponent is negative if $u^2 > 1$. If one assumes that the estimation (29) is true for $u^2 \le 1$ (which is not evident) than the Lyapunov exponent becomes positive. This analysis is confirmed by the direct calculation of the Lyapunov exponent by the formula (28), according to which its magnitudes become negative beginning from the generalized parameter value of $u \approx 0.5$.

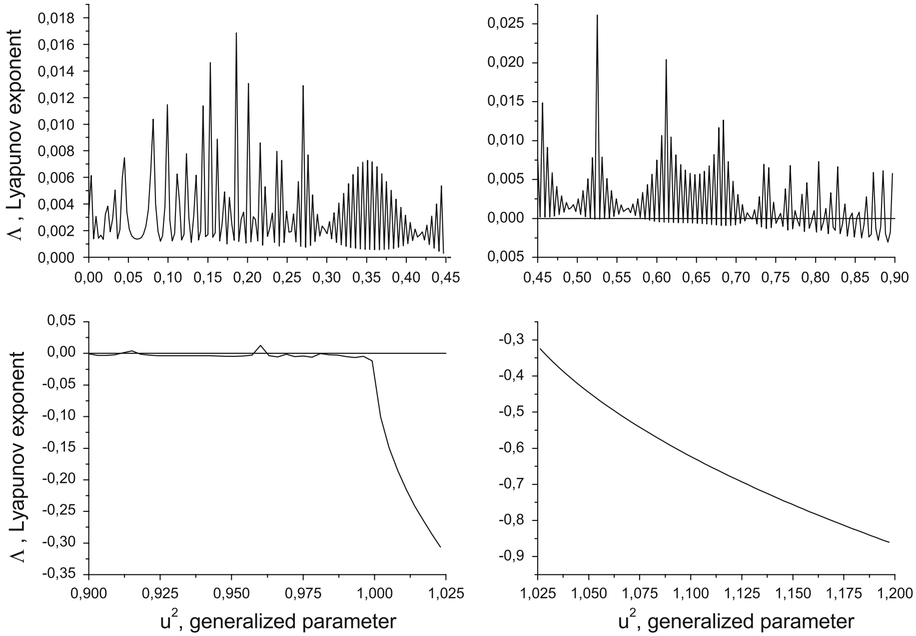

Fig. 10 Dependence of the Lyapunov exponent on the generalized parameter

Figure 10 shows that the Lyapunov exponent becomes positive for $u \leq 0.6$. In the region $\sim 0.6 < u < 1$ there is a set of intermittent intervals of chaotic and non-chaotic behaviour of the Lyapunov exponent. For $u > 1$ the Lyapunov exponents become strictly negative.

5 Conclusion

The three simplest linear physical models where the chaotic dynamics can be realized are investigated: three linear interacting oscillators, electromagnetic wave scattering in a double periodical inhomogeneous medium occupying half-space and an electromagnetic field in homogeneous linear dielectric medium which permittivity is modulated in time.

It is shown that these linear systems and some their characteristics can have chaotic behaviour. It is worth to note that these types of models are often used in the description of various physical processes.

One can conclude that the change of the dependent variables can transform the initially linear equations into nonlinear ones that can have chaotic solutions. The chaotic behaviour may reflect also the fact that the new variables themselves satisfy nonlinear equations.

Chaotic dynamics generated by linear systems allows to reveal unknown features of such systems that may be useful for their better and more complete understanding. Moreover, methods of statistical physics can be used for the investigation of the system characteristics in the regime of dynamical chaos.

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