Time-spatial drift of decelerating electromagnetic pulses

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Abstract: A time dependent electromagnetic pulse generated by a current running laterally to the direction of the pulse propagation is considered in paraxial approximation. It is shown that the pulse envelope moves in the time-spatial coordinates on the surface of a parabolic cylinder for the Airy pulse and a hyperbolic cylinder for the Gaussian. These pulses propagate in time with deceleration along the dominant propagation direction and drift uniformly in the lateral direction. The Airy pulse stops at infinity while the asymptotic velocity of the Gaussian is nonzero.

OCIS codes: (050.1940) Diffraction; (260.2110) Electromagnetic optics; (350.5500) Propagation.

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#190232 - $15.00 USD Received 9 May 2013; revised 2 Jul 2013; accepted 3 Jul 2013; published 12 Jul 2013

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1. Introduction

Intensive theoretical and experimental investigations of Airy beams are motivated by their unusual features (non-diffractive propagation, accelerating motion, and self-healing). A solution to the Schrodinger equation in the form of a non-spreading accelerating Airy wave function found by Berry and Balazs in 1979 [1] inspired Siviloglou and Christodoulides to put forward the concept of electromagnetic accelerated Airy beams [2, 3]. These seminal publications with theoretical formulations based on the paraxial approximation to the wave equation and experimental confirmation were followed by many works on the Airy beam properties, for example [4–7] (and citations therein). These investigations generate very interesting applications, some of which have been already realised [8–14]. In the majority of these investigations time is eliminated from the paraxial equation by using the predominant time harmonic dependence $E=\tilde{F}(x,y,z)e^{i(\omega t-kx)}$ for a wave propagating along the x axis. In this case a parabolic dependence exists between the lateral (y,z) and the longitudinal x coordinates in $\tilde{F}(x,y,z)$. The parabolic dependence between time and the longitudinal coordinate with the leading dependence $\sim e^{i\omega t}$ is also considered in a number of publications. It is shown in [2] that a circular symmetric input field with the temporal behaviour according to the Airy function keeps this symmetry at later times propagating with time acceleration. Wave functions constructed in time domain using the Fourier transform are considered in [4, 15–21]. Spatiotemporal Airy light bullets as the solution to a paraxial equation in a dispersive medium are investigated in [16, 17]. The temporal analysis of the Airy pulses in dispersive and nonlinear medium reveals such phenomena as generation of solitons [20].

A fundamental remark concerning the causality effects of the phenomenon is made in [21]. It is noted that the spatial and temporal accelerations are physically qualitatively different. According to the authors' mathematical formulation of the problem [21], if the accelerated motion is considered in time, the solution can give accelerating or decelerating movement and requires the backward flow of time on a part of the trajectory. Some physical ideas for overcoming this problem are proposed. However, these ideas look rather artificial because they are built on solutions to homogeneous paraxial equation without taking into account neither initial (boundary) conditions nor sources creating the field.

Here we add external sources to the problem and show that in this formulation no exotic suggestions are needed and no backward flow of time appears. We do not make any assumptions about the temporal dependence of the field. We consider this process in paraxial approximation [22–24] assuming the dominant propagation along the selected x axis, $E=F(t,x,y,z)e^{-ikx}$, and formulate the master equation in this approximation. We derive its solution by a rigorous method of the Green's function that allows constructing other decelerating pulses (not Airy), the Gaussian is given as an example.

2. The master equation

To investigate the acceleration of electromagnetic pulses in time we consider a problem rigorously with only one assumption that is of paraxial approximation. It means that the field takes the dependence on the longitudinal coordinate x dominant, $E=F(t,x,y,z)e^{-ikx}$, $k>0$, such that the well-known paraxial approximation of a slow varying envelope $\left|F_{xx}^{''}\right|<<\left|2ik_xF_x^{'}\right|$

is fulfilled. We do not consider free waves, but a more practical situation when the field is created by an external electric current $j(t,x,y,z)$. This field is described by an inhomogeneous wave equation which produces the master paraxial equation for the envelope F with no predefined temporal dependence

$$-2ik\frac{\partial F}{\partial x}-k^2F+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}-\mu_0\mu\sigma\frac{\partial F}{\partial t}=f. \tag{1}$$

Here the right-hand side $f=\mu_0\mu\frac{\partial j}{\partial t}e^{ikx}$ represents the source current, $v=1/\sqrt{\varepsilon_0\varepsilon\mu_0\mu}$ is the velocity of light, ε_0 and μ_0 are the permittivity and the permeability of vacuum, ε and μ are the permittivity and the permeability of the nondispersive medium where the field is created. The medium dissipation is determined by the conductivity σ.

The solution to Eq. (1) is given by the convolution $F=G\otimes f$ of the right hand side f with the Green's function G of this equation [25]. This function G is found as a solution to Eq. (1) but with the delta-function on the right-hand side:

$$G=\frac{-(1-i)v}{8\pi x}\sqrt{\frac{k}{\pi x}}\theta(x)e^{i\frac{k}{2}x+i\frac{kv^2}{2x}[(t+i\frac{\mu_0\mu\sigma}{2k}x)^2-y^2-z^2]}. \tag{2}$$

It does not contain retardation and describes the propagation of a disturbance with the infinite velocity as it is inherent to solutions of parabolic type equations [26]. This means that the processes described by this function do not contain the cause-effect relationship.

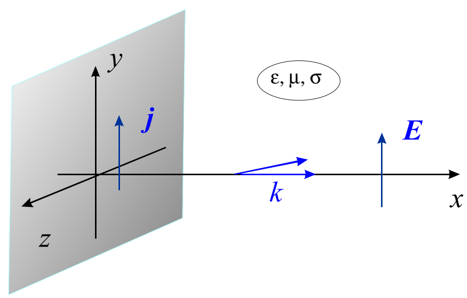

Fig. 1. The geometry of the problem: an electric current $\mathbf{j}$ in a lateral plane located at x_0 excites a pulse $\mathbf{E}$ propagating in the perpendicular direction.

Writing the current in the spectral representation over plane waves in the lateral source plane $x=x_0$, which is perpendicular to the direction of the radiation propagation, Fig. 1,

$$j=j_0\delta(x-x_0)\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)e^{i\omega(t-q_1y-q_2z)}d\omega \tag{3}$$

and executing the convolution we obtain the spectral representation of the radiated field

$$E=j_0\frac{\mu_0\mu}{2ik}\theta(x-x_0)e^{i(x-x_0)\frac{q^2-k^2}{2k}}\frac{\partial}{\partial t}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\Phi(\omega)e^{-i\omega^2\frac{m(x-x_0)}{2kv^2}+i\omega(t+i\frac{\mu_0\mu\sigma}{2k}(x-x_0)-q_1y-q_2z)}, \tag{4}$$

where $m=1-q^2v^2$, $q^2=q_1^2+q_2^2$. The coefficient m shows the degree of paraxial approximation. If $m=0$ ($q=1/v$) then the squared frequency in the integrand exponent in

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Eq. (4) disappears. However, in this case the parabolic approximation is inapplicable. The pure case of parabolic approximation is realised when $m \to 1$ ($q \to 0$).

The space time structure of this field is determined by the current's spectral function and, following the Green's function (2), the radiated field (4) does not contain retardation too. Therefore, it does not describe the relationship of cause-effect.

3. Decelerating Airy pulse

The current provided by the odd spectrum function $\Phi(\omega) = \exp(i(\omega T + i\alpha)^3 / 3)$ in (3) corresponds to the pulse running in the plane $x = x_0$ and described by the Airy function [27],

$$j = \delta(x - x_0)\mathrm{Ai}\left((t - q_1 y - q_2 z)/T\right) e^{\alpha(t - q_1 y - q_2 z)/T} / T. \tag{5}$$

Here, T is a normalized parameter and the parameter α is introduced for ensuring energy finiteness of the source, the idea proposed in [2] for a problem with the harmonic temporal dependence of the phenomenon.

The calculation of the integral in Eq. (4) gives the radiated wave with the envelope described by the Airy function but with the quadratic argument in contrast to the linear one in Eq. (5):

$$\begin{aligned} E(t,x) &= \frac{i\mu_0\mu}{2kT} j_0 e^{-ik(x-x_0)/2}\theta(x-x_0)\varphi(x) \\ &\times \frac{\partial}{\partial t}\left\{ \exp\left[i\left((\tau + i\frac{\mu_0\mu\sigma}{2k}(x-x_0))\right)\frac{1}{T}(m\frac{x-x_0}{2kv^2T^2} - i\alpha) \right] \right. \\ &\left. \times \mathrm{Ai}\left[\left(\overline{t} + i\frac{\mu_0\mu\sigma}{2k}(x-x_0) \right)\frac{1}{T} - \left(m\frac{x-x_0}{2kv^2T^2} \right)^2 + 2i\alpha m\frac{x-x_0}{2kv^2T^2} \right] \right\} \end{aligned} . \tag{6}$$

The function $\varphi(x) = i\frac{2}{3}\left(m\frac{x-x_0}{2kv^2T^2}\right)^2 - 2\alpha\left(m\frac{x-x_0}{2kv^2T^2}\right)^2 + i\alpha^2 m\frac{x-x_0}{2kv^2T^2}$ in (6) depends only on the longitudinal coordinate *x*. The retarded time $\overline{t} = t - q_1 y - q_2 z$ depends on the instant time and the lateral coordinates.

The obtained solution shows that the Airy pulse of the current running in the source plane lateral to the radiation direction radiates the Airy pulse propagating along a complex trajectory. Specifically, the trajectory of the pulse envelope lies on the surface of a parabolic cylinder in space time coordinates (t, x, y, z), Fig. 2. The equation of this surface is derived by equating the real part of the Airy function argument in (6) to some constant a

$$(t - q_1 y - q_2 z)\frac{1}{T} - \left(m\frac{x-x_0}{2kv^2T^2}\right)^2 = a. \tag{7}$$

The pulse envelope moves on this surface decelerating along the longitudinal axis x and drifts uniformly with time in the lateral direction. The projection of the parabolic cylinder onto the plane (t,x) illustrates the trajectories of the envelope in space and time. The vertex of the parabolic trajectory is situated at the source point and each trajectory goes away from the source ($x_0 = 0$) with time. Therefore, the problem with the backward flow of time does not appear.

#190232 - \$15.00 USD Received 9 May 2013; revised 2 Jul 2013; accepted 3 Jul 2013; published 12 Jul 2013

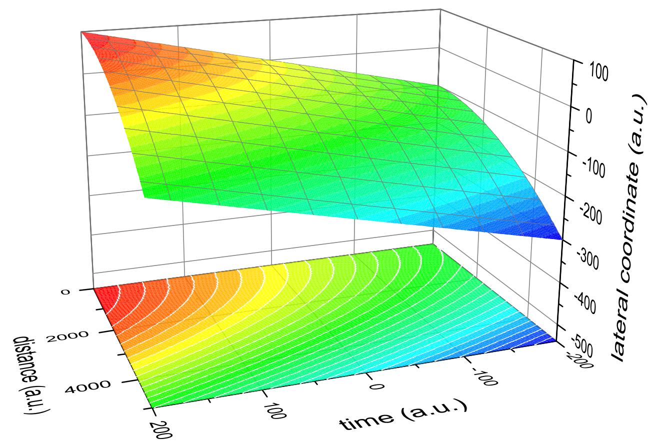

Fig. 2. The surface of the parabolic cylinder in the space time coordinates along which the Airy pulse propagates.

By differentiating Eq. (7) with respect to time we obtain the dependence between the components of the complete envelope velocity

$$q_1\dot{y}+q_2\dot{z}+\frac{(x-x_0)m^2}{2k^2v^4T^3}\dot{x}=1. \tag{8}$$

At each fixed point *x*, $\dot{x}=0$, the velocity in the transverse direction is constant $q_1\dot{y}+q_2\dot{z}=1$. The velocity of the envelope movement in the longitudinal direction is derived from Eq. (8) assuming that *y* and *z* are constants. This gives the envelope velocity along the propagation direction $\dot{x}=2k^2v^4T^3m^{-2}/(x-x_0)$ as a function of the distance from the source or as a function of time $\dot{x}=kv^2T^{3/2}m^{-1}/\sqrt{t-aT}$. Here the constant a is as in Eq. (7). This velocity tends to zero with time as well as with the distance from the source. The acceleration of this movement $\ddot{x}=-\dot{x}^2/(x-x_0)$ is negative everywhere in the region of the pulse existence, $x-x_0>0$, and it also tends to zero with the distance from the source confirming the decelerating character of the movement.

4. Decelerating Gaussian

The source of the even spectrum function $\Phi(\omega)=\exp(-(\omega T)^2/4w^2)$ in Eq. (3) corresponds to the source current of the Gaussian form running in the transverse plane

$$j=j_0\delta(x-x_0)wT^{-1}\pi^{-1/2}e^{-w^2(t-q_1y-q_2z)^2/T^2}. \tag{9}$$

The field radiated by this source is the well known running Gaussian

$$E=\frac{1}{2ik}j_0\mu_0\mu e^{-ik(x-x_0)/2}\frac{w\theta(x-x_0)}{2T\sqrt{\pi}}\frac{1}{\sqrt[4]{1+m^2u^2(x-x_0)^2}}\frac{\partial}{\partial t}(AR), \tag{10}$$

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$$\text{where } A=\exp\left(-\frac{w^2}{T^2}\frac{\bar{t}^2+\bar{t}\frac{\mu_0\mu\sigma}{k}mu(x-x_0)^2-\left(\frac{\mu_0\mu\sigma}{2k}\right)^2(x-x_0)^2}{1+m^2u^2(x-x_0)^2}\right),$$

$$R=\exp\left(i\frac{muw^2(x-x_0)\left[\bar{t}^2-\left(\frac{\mu_0\mu\sigma}{2k}\right)^2(x-x_0)^2-\bar{t}\frac{\mu_0\mu\sigma}{uk}\right]}{T^2(1+m^2u^2(x-x_0)^2)}-\frac{i}{2}\arctan(mu(x-x_0))\right),$$

$\bar{t}=t-q_1y-q_2z$ and $u=\dfrac{2w^2}{kv^2T^2}$.

This expression is significantly simplified in the case of a non-dissipative medium, $\sigma=0$:

$$E=\frac{1}{2ik}j_0\mu_0\mu e^{-ik(x-x_0)/2}\frac{w\theta(x-x_0)}{2T\sqrt{\pi}}\frac{1}{\sqrt[4]{1+m^2u^2(x-x_0)^2}}\times$$
$$\frac{\partial}{\partial t}\exp\left[-\frac{w^2}{T^2}\frac{\bar{t}^2}{1+m^2u^2(x-x_0)^2}\right]\exp\left[i\left(\frac{w^2}{T^2}\frac{\bar{t}^2mu(x-x_0)}{1+m^2u^2(x-x_0)^2}-\frac{1}{2}\arctan(mu(x-x_0))\right)\right]. \quad (11)$$

Equating the argument of the second exponent in (11) to some constant a, as in Eq. (7), we obtain the equation for the surface of the hyperbolic cylinder $(t-q_1y-q_2z)^2a^{-1}-m^2u^2(x-x_0)^2=1$, Fig. 3, on which the pulse envelope moves. The projection of the hyperbolic cylinder onto the plane (t,x) illustrates the decelerating motion of the Gaussian envelope along the hyperbolic trajectories, but its velocity changes in different manner comparing with the Airy pulse.

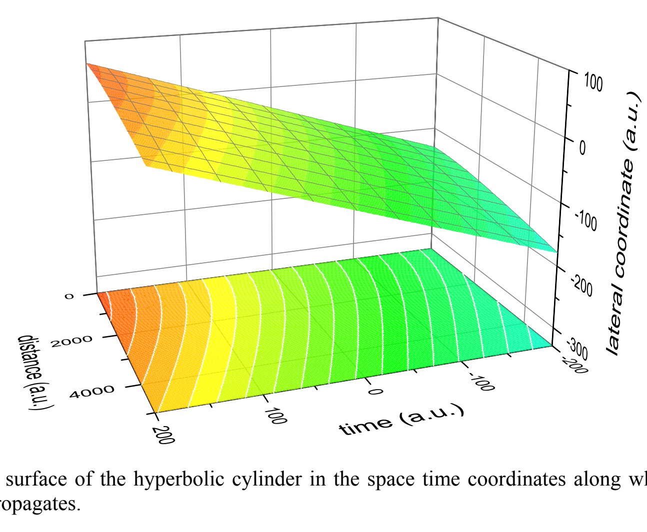

Fig. 3. The surface of the hyperbolic cylinder in the space time coordinates along which the Gaussian propagates.

The dependence between the longitudinal and the transverse velocities on this surface is obtained by differentiating this equation

#190232 - $15.00 USD Received 9 May 2013; revised 2 Jul 2013; accepted 3 Jul 2013; published 12 Jul 2013

$$(1-q_1\dot{y}-q_2\dot{z})(t-q_1y-q_2z)/a-\dot{x}m^2u^2(x-x_0)=0. \qquad (12)$$

In the lateral direction (for a fixed coordinate x) the envelope movement, as in the case of the Airy pulse, is uniform and its velocity is constant: $q_1\dot{y}+q_2\dot{z}=1$. For given values of y and z the envelope movement is decelerating along the longitudinal axis x and its velocity changes as

$$\dot{x}=\frac{t}{a\,m^2u^2(x-x_0)}=\frac{\sqrt{1+m^2u^2(x-x_0)^2}}{\sqrt{a}\,m^2u^2(x-x_0)}=\frac{t}{\sqrt{a}\,mu\sqrt{t^2-a}}. \qquad (13)$$

In contrast to the Airy pulse this velocity asymptotically tends to the nonzero value $\dot{x}_\infty=1/\sqrt{m^2u^2a}$ with time as well as with the distance from the source.

5. Conclusions

A time dependent electromagnetic field in paraxial approximation generated by a current running in a plane transverse to the propagation direction of the electromagnetic pulse is considered. This field is described by a differential equation of a parabolic type with a source on the right-hand side. The radiation of such source in the form of Airy and Gaussian pulses is investigated. The pulse envelope moves in the (t,x,y,z) coordinates on the parabolic cylinder surface for the Airy pulses and the hyperbolic cylinder for the Gaussian. It is shown that these pulses propagate in time with deceleration along the dominant propagation direction and drift uniformly in time and in the lateral direction. The velocity of the envelope tends to zero with time and distance from the source for the Airy pulse and to an asymptotic nonzero value for the Gaussian. The considered statement of the problem does not require the backward flow of time.