Decelerating non-diffractive electromagnetic Airy pulses

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***Abstract*— The existence of electromagnetic pulses in time domain with the Airy function envelope is shown. The pulses satisfy an equation similar to the Schrodinger equation but in which the time and space variables play opposite roles. The pulses are generated by an Airy time varying electromagnetic field at a source point and propagate in vacuum preserving their shape and magnitude. The motion is according to a quadratic law with the velocity changing from infinity at the source point to zero in infinity. These one-dimensional results are extended to the 3D+time case when a similar Airy-Bessel pulse is excited by the field at an plane aperture perpendicular to the direction of the pulse propagation. The same behaviour of the pulses, the non-diffractive preservation and their deceleration, is found.**

I. INTRODUCTION

An Airy beam or Airy wave packet is a wave described by the Airy function [1]. The Airy beams are characterised by very special properties: they are *non-diffractive* (remain invariant during propagation) and *accelerating* (increase their envelope velocity with time) [2-8]. Recently there has been active development in the theory and experimental applications of *optical* Airy beams [9-13].

The detailed analysis of the mathematical aspects as well as physical interpretation of electromagnetic Airy beams is done by considering the wave as a function of *spatial coordinates only* and assuming that their time dependence is harmonic, $\sim \exp(i\omega t)$, [2-8]. Yet, the idea of electromagnetic Airy beams comes from the analogy of the paraxial equation describing these beams with the time dependent Schrodinger equation [14], where the time variable is replaced with a spatial coordinate. The solution of the Schrodinger equation produces time dependent Airy wave packets in free space [14]. Their features such as the diffraction free form and continuous acceleration has been explained on the basis of the semi-classical approximation. As for the time dependent solution of the three-dimensional electromagnetic problem, the possibility of the existence of non-diffractive Bessel (not Airy) waves has been pointed out in [2, 11]. However, the three-dimensional solutions to the paraxial equations containing the time derivative do not include the parabolic variable responsible for the accelerating feature of the beams.

Therefore, it is important to investigate the *explicitly time dependent* solutions of the electromagnetic problem in the form of an Airy pulse and deduce whether it possesses the unique features described above. We show that it is not only possible to find the Airy pulse solution starting from the first principles, rather than by exploiting the analogy with the paraxial equation, but also that the obtained beam has the same property of non-diffractive propagation and velocity change without any external influences (in vacuum). There are, however, important conceptual differences that lead to the pulse deceleration, rather than acceleration as in quantum mechanics.This document is a template. An electronic copy can be downloaded from the conference website.

II. MAINE EQUATIONS

We consider here the role of the time variable in the solution of a 'paraxial' equation including explicit presence of time. We start with the wave equation, followed from the Maxwell equation,

$$\frac{\partial^2}{\partial z^2}E(t,z)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(t,z)=0 \qquad (1)$$

which describes the electric field of a wave propagating along the z axis. Substitution of the field in the form $E(t,z)=F(t,z)e^{\pm ikz}$, $k=\omega/c$ and under the assumption that $\left|F_{zz}^{''}\right| \ll \left|2ikF_{z}^{'}\right|$, typical for the paraxial approximation [15], the wave equation is reduced to the form

$$\mp i\frac{\partial}{\partial\xi}F+\frac{1}{2}\frac{\partial^2}{\partial\tau^2}F+\frac{1}{2}\kappa^2F=0\,, \qquad (2)$$

where the normalised dimensionless variables are $\xi=z/(kc^2t_0{}^2)$ and $\tau=t/t_0$ with t_0 being the temporal scale and the dimensionless parameter $kct_0=\kappa$. Comparing this equation with the commonly considered spatial paraxial equation in the x, z coordinates

$$i\frac{\partial}{\partial\xi}\Phi+\frac{1}{2}\frac{\partial^2}{\partial s^2}\Phi=0 \qquad (3)$$

we see that the longitudinal spatial variables ξ are the same and the transverse variable $s=x/x_0$ $(x_0=ct_0)$ corresponds to the temporal variable τ in (2). The equation (3) is considered in the literature as the analogue to the Schrodinger equation [11]

$$\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)+i\hbar\frac{\partial}{\partial t}\Psi(x,t)=0 \qquad (4)$$

from which the Airy wave packet originated in [2] if the temporal variable t in (4) is replaced by the longitudinal variable z (ξ in (3)). Thus, the variable z (ξ) along which an electromagnetic wave propagates plays the role of time in the electromagnetic phenomenon. As it was shown in [14] equation (4) has a solution in the form of a non-spreading wave packet with the envelope as the Airy function

$$\Psi(x,t)=\mathrm{Ai}\left[\frac{B}{\hbar^{2/3}}\left(x-\frac{B^3t^2}{4m^2}\right)\right]exp\left[i\frac{B^3}{2m\hbar}\left(tx-\frac{B^3t^3}{6m^2}\right)\right]. \qquad (5)$$

This function describes the accelerating wave packet which moves uniformly with the velocity $\dot{x}=B^3t/2m^2$ and the constant acceleration $\ddot{x}=B^3/2m^2$. Contrary to the equation (3), which describes a beam harmonically oscillating in time, the function (5) represents the pulse with a complicated time varying envelope enclosed in the Airy function. The Airy function in the solution to (3)

$$\Phi=\mathrm{Ai}[s-\frac{1}{4}\xi^2]\exp\left\{i\left[-\frac{s\xi}{2}+\frac{1}{12}\xi^3\right]\right\} \qquad (6)$$

describes the inhomogeneous distribution with respect to the spatial coordinates s and ξ of the wave paraxial propagating along the z axis but with harmonic temporal variation $E=\Phi(x,z)e^{ikz-i\omega t}$.

Our equation (2), derived from the first principle rather than by the analogy with the Schrodinger equation, shows that the roles of the time and space variables in the electromagnetic time paraxial equation (2) are opposite to those of the Schrodinger equation (4). This destroys the analogy between the equations (2) and (4) and, therefore, the direct correspondence between the time and space variables of the Schrodinger equation and the space variables of the spatial paraxial equation (3). Thus, we need to solve the equation (2) in order to find the time-spatial pulse originating from it.

III. Airy Pulses

The solution to the equation (2) can be constructed following the procedure described in [8]. The sought function is represented as $F=W(\eta)e^{i\Theta(\eta,\xi)}$, where $W(\eta)$ and $\Theta(\eta,\xi)$ are real functions of the argument ξ and the quadratic variable $\eta=-a\tau+\tau_0-g\xi^2/4+b\xi$. The parameters a, τ_0, g, and b in η allow changing the model. This representation leads to the equation for the envelope $W''(\eta)-(g\eta/a^4)W=0$ and to the equation for the phase

$$\Theta(\eta,\xi)=\pm\frac{1}{a^2}\left(-\frac{g}{2}\xi+b\right)\eta \mp\frac{1}{2a^2}\left(\frac{g^2}{12}\xi^3-\frac{g^2b}{2}\xi^2+(b^2+a^2\kappa^2)\xi\right). \qquad (7)$$

The parameter $a=\pm1$ determines the movement forward or backward along the spatial axes but both equations do not depend on it. The parameter g defines only the scale factor in variables normalisation and can be taken $g=1$. The result is the Airy equation $W''(\eta)-\eta W(\eta)=0$ with the solution as the Airy function

$$W(\eta)=\mathrm{Ai}[-\left(\frac{1}{2}\xi-b\right)^2-a\tau+\tau_0+b^2]. \qquad (8)$$

Choosing $a=-1$ we obtain the solution to the equation (2) as

$$F(\tau,\xi)=\mathrm{Ai}[-\left(\frac{1}{2}\xi-b\right)^2+\tau+\tau_0+b^2]\times \exp\left\{\pm i\left[-(\tau+\tau_0+b^2)\left(\frac{1}{2}\xi-b\right)+ \frac{1}{12}\xi^3-\frac{1}{2}b\xi^2+\frac{(2b^2-\kappa^2)}{2}\xi-b^3\right]\right\}. \qquad (9)$$

This function satisfies the boundary condition $F(\tau,\xi=2b)=\mathrm{Ai}[\tau+\tau_0+b^2]\exp\left\{\pm i\left[-b(\kappa^2+b^2/3)\right]\right\}$ that can be interpreted as a time varying source located at the point $\xi=2b$. Therefore, the solution (9) describes the propagation of this source radiation

$$E=F(\tau,\xi)e^{\pm i\kappa^2\xi}=\mathrm{Ai}\left[-\left(\frac{\xi}{2}-b\right)^2+\tau+\tau_0+b^2\right]\times \exp\left\{\pm i\left[-(\tau+\tau_0+b^2)\left(\frac{1}{2}\xi-b\right)+ \frac{1}{12}\xi^3-\frac{b}{2}\xi^2+\frac{(2b^2+\kappa^2)}{2}\xi-b^3\right]\right\}. \qquad (10)$$

This field is uniquely defined in the half-space $\xi\geq 2b$, Fig.1.

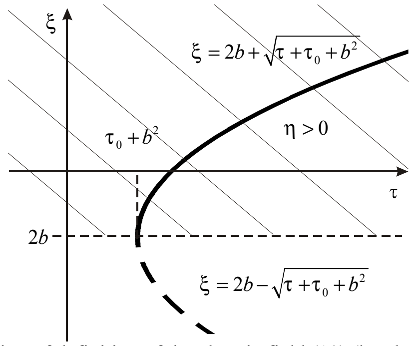

Fig. 1. The region of definition of the electric field (10) (hatched region).

Starting from the source point, at which the field time variation is given by the formula $E(\tau,\xi=2b)=\mathrm{Ai}\left[\tau+\tau_0+b^2\right]\exp\left[\pm ib(\kappa^2-b^2/3)\right]$, the field profile propagates according to the quadratic law $\tau+\tau_0-(\xi/2-b)^2+b^2=const$ preserving its form. Fig.1 illustrates the lines of propagation of the field equal values determined by the parabola $\tau+\tau_0-\xi^2/4+b\xi=const$ (one of the branches for $const=0$ is shown using the solid line in the figure). The quadratic variable η is positive inside the region bounded by the parabola and negative outside of it. It determines where the steep front of the Airy pulse is directed along the movement. The velocity of this movement decreases with distance $\dot{\xi}=2/(\xi-2b)$, therefore the acceleration $\ddot{\xi}=-4/(\xi-2b)^3$ is negative. Such a slowing motion leads to a complete stop as its velocity and acceleration tend to zero at the infinite distance from the source, Fig. 2.

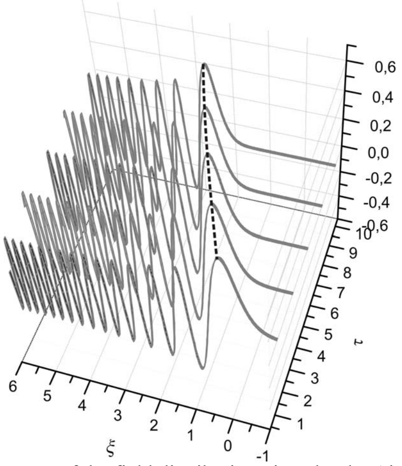

Fig. 2. The movement of the field distribution given by the Airy function envelope in (10) (the magnitudes of this envelope is shown on the vertical axes).

The considered Airy pulses are of little practical importance because of their infinite energy. To overcome this deficiency it was suggested in [2, 3] to consider the exponentially decaying version at the input of the system. Following this suggestion we consider a different boundary condition

$$\begin{gathered}F(\tau,\xi=2b)=\mathrm{Ai}[\tau+\tau_0+b^2]\times\\ \exp\left\{\pm i\left[-b(\kappa^2+b^2/3)\right]+\alpha(\tau+\tau_0+b^2)\right\}\end{gathered} \tag{11}$$

for obtaining the pulse with finite energy. To solve the equation (2) with the boundary condition (11) we represent the solution as the Fourier transform

$$F(\tau,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu e^{i\nu\tau}\bar{F}(\nu,\xi)\,. \tag{12}$$

Using this representation and the condition (11) we can write the boundary condition for the inverse Fourier transform

$$\begin{gathered}\bar{F}(\nu,2b)=\int_{-\infty}^{\infty}d\tau e^{-i\nu\tau}\mathrm{Ai}[\tau+\tau_0+b^2]\times\\ \exp\left\{\pm i\left[-b(\kappa^2+b^2/3)\right]+\alpha(\tau+\tau_0+b^2)\right\}=\\ \exp[i\nu(\tau_0+b^2)\mp ib(\nu^2+b^2/3)+i(\nu+i\alpha)^3/3]\end{gathered}, \tag{13}$$

where the Fourier representation of the Airy function is used [1].

The equation for the Fourier transform following from (2)

$$\mp i\frac{\partial}{\partial\xi}\bar{F}(\nu,\xi)-\frac{\nu^2-\kappa^2}{2}\bar{F}(\nu,\xi)=0 \tag{14}$$

has the solution satisfying the condition (11)

$$\begin{gathered}\bar{F}(\nu,\xi)=\exp\left[i\nu(\tau_0+b^2)\mp ib(\kappa^2+b^2/3)\pm\right.\\ \left.i(\nu^2-\kappa^2)(\xi/2-b)+i(\nu+i\alpha)^3/3\right]\end{gathered}. \tag{15}$$

Applying the Fourier transform (12) we obtain the electric field of the pulse

$$\begin{gathered}E(\tau,\xi)=\mathrm{Ai}[\tau+\tau_0+b^2-(\xi/2-b)^2\mp i2\alpha(\xi/2-b)]\times\\ \times\exp\left\{\alpha^3/3\mp ib(\kappa^2+b^2/3)+\alpha(\tau+\tau_0+b^2)-\right.\\ 2\alpha(\xi/2-b)^2\mp i(\tau+\tau_0+b^2)(\xi/2-b)\pm\\ \left.i2(\xi/2-b)^3/3\pm i\kappa^2(\xi/2+b)\pm\alpha^2(\xi/2-b)\right\}\end{gathered}. \tag{16}$$

Fig. 3 shows the envelope distribution in (16) for two time moments. The steep front of the Airy envelope, corresponding to nearly zero moment at the source point, comes off the source leaving the space free of the field.

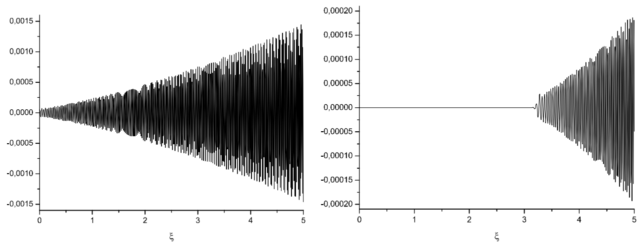

Fig. 3. The field distribution of (16) for two time moments: $\tau=0$ (left) and $\tau=100$ (right), $\kappa=500$, $b=-30$ for both moments.

IV. 3D+T Airy pulses

The described model can be straightforwardly extended to the 3D+time case. If a phenomenon has the cylindrical symmetry then the wave equation in a cylindrical system of coordinates takes the form

$$\begin{gathered}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right)E(t,\rho,\varphi,z)-\\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(t,\rho,\varphi,z)=0\end{gathered}. \tag{17}$$

Representing the solution in the form

$$E(t,\rho,z)=R(\rho)F(t,z)e^{im\varphi\pm ikz}, \tag{18}$$

where the transverse and the longitudinal variables are separated, and taking the same paraxial approximation of slow changing with the longitudinal coordinate we reduce the equation (17) to

$$\left[\pm i\frac{\partial}{\partial\xi}-\frac{1}{2}\frac{\partial^2}{\partial\tau^2}-\frac{\kappa^2}{2}+\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}\right)\right]R(\rho)F(\tau,\xi)=0 \tag{19}$$

where $r = \rho / ct_0$. Bearing in mind the Bessel equation $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + (\mu^2 - \frac{m^2}{r^2})\right) R(\rho) = 0$ we can take the radial dependence as the Bessel function $R(r) = J_m(\mu r)$ that reduces (19) to the equation similar to (2)

$$\left(\mp i\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial^2}{\partial \tau^2} + \frac{\kappa^2 + \mu^2}{2}\right) F(\tau,\xi) = 0. \qquad (20)$$

The solution to this equation is obtained from (16) by merely substituting $\kappa^2 \to \kappa^2 + \mu^2$. This solution describes the radiation from an aperture with the following distribution of the field

$$\begin{aligned} &E(\tau, r, \xi = 2b) = J_m(\mu r)\mathrm{Ai}[\tau + \tau_0 + b^2]\times \\ &\exp\left\{\pm i\left[-b(\kappa^2 + 1 + b^2/3)\right] + \alpha(\tau + \tau_0 + b^2)\right\}. \end{aligned} \qquad (21)$$

This radiation propagates normally to the aperture and slows down with distance to stopping:

$$\begin{aligned} &E(\tau, r, \xi) = J_m(\mu r)\mathrm{Ai}[\tau + \tau_0 + b^2 - (\xi/2 - b)^2 \mp i2\alpha(\xi/2 - b)]\times \\ &\times\exp\left\{\alpha^3/3 \mp ib(\kappa^2 + \mu^2 + b^2/3) + \alpha(\tau + \tau_0 + b^2) - \right. \\ &2\alpha(\xi/2 - b)^2 \mp i(\tau + \tau_0 + b^2)(\xi/2 - b) \pm \\ &\left. i2(\xi/2 - b)^3/3 \pm i(\kappa^2 + \mu^2)(\xi/2 + b) \pm \alpha^2(\xi/2 - b)\right\}. \end{aligned} \qquad (22)$$

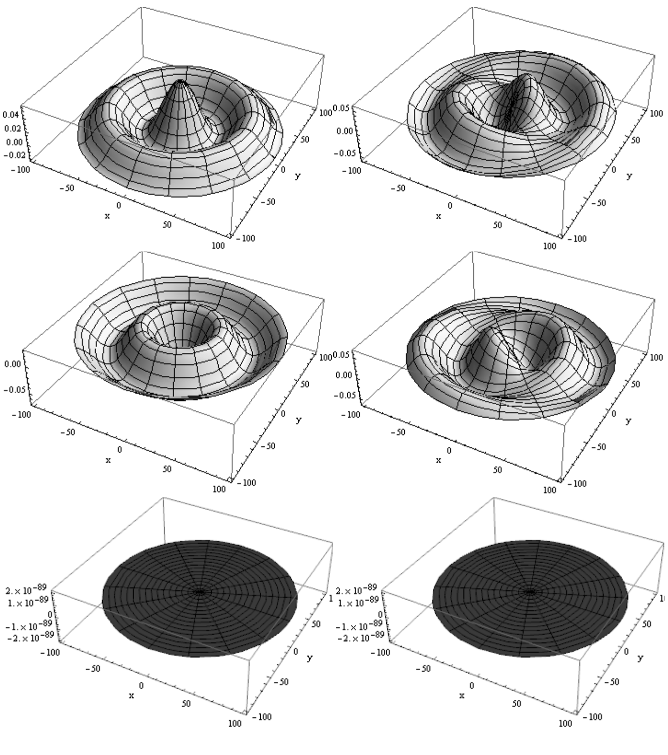

Fig. 4. The evolution of the electric field (22) at $\tau = 0$ (top), $\tau = 10$, and $\tau = 100$ (bottom) for *m*=0 (left column) and *m*=1 (right column)

Fig. 4 illustrates the snapshots of the field transverse distribution at the point $\xi = 10$ ($0 \le r \le 100$, $0 \le \varphi \le 2\pi$) at three instants. These distributions correspond to the radial symmetry (*m*=0, the left column) and to one variation in the azimuthal angle (*m*=1, the right column) at the aperture ($\mu = 1$ in both cases). The snapshots at $\tau = 0$ and $\tau = 10$ show the pronounced inhomogeneous distributions of the field whereas at $\tau = 100$ the field is essentially zero. This reproduces the behaviour of the one-dimensional problem in that the pulse comes off the source and leaves the space free of the field, Fig. 3, right.

V. Conclusion

In conclusion, we derived the time dependent electromagnetic Airy pulses that satisfy the 'paraxial' equation similar to the Schrodinger equation in which the time and space variables interchange their roles. The solution to the electromagnetic equation is the Airy pulse which propagates with deceleration along its trajectory and stops at the infinite distance from the source. In the 3D case the similar Airy-Bessel pulse occurs when the radiation is excited by the field at the aperture which is perpendicular to the direction of the pulse propagation. If the field at the aperture is distributed as $J_m(\mu r)$ then the transverse distribution defined by the Bessel function propagates from the aperture preserving its form at all distances.

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