Hurst's Index and Complexity of Wave in Modulated Dielectric Medium

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ABSTRACT
Using an exactly solvable model a process of modulation of an electromagnetic field in a time-varying medium is investigated. A correlation between the Hurst's index of a transformed electromagnetic signal and its complexity is considered for the first time.
Keywords: Electromagnetic transients, time-varying medium, intermittency, complexity.

1. INTRODUCTION

Parametric phenomena in active media have been attracting much attention for a long time in connection with the generation and amplification of electromagnetic waves or with the time variation of the medium parameters. In the systems with distributed parameters main features of the wave transformation by the medium nonstationarity can be revealed when a simple law changes the medium parameters such that an exact solution of the problem can be constructed. In this paper the electromagnetic wave transformation in a medium with parameters that undergo changes in a form of a finite packet of periodic rectangular pulses is considered. Regularity of the transformation is estimated by two characteristics, the Hurst's index [1] and the complexity [2].

2. WAVE TRANSFORMATION UNDER MEDIUM MODULATION

We consider an unbounded dielectric dissipative medium, the permittivity and conductivity of which are modulated according to the law of a finite packet of N rectangular periodic pulses. This modulation is given by

$$\begin{aligned} \varepsilon(t) &= \varepsilon + (\varepsilon_1 - \varepsilon)\sum_{k=1}^{N}\left\{\theta(t-(k-1)T) - \theta(t-T_1-(k-1)T)\right\} \\ \sigma(t) &= \sigma_1\sum_{k=1}^{N}\left\{\theta(t-(k-1)T) - \theta(t-T_1-(k-1)T)\right\} \end{aligned} \qquad (1)$$

Here, $\theta(t)$ is the Heaviside unit function, T is the duration of the period of the parameters change, T_1 is the duration of the disturbance interval, in which the medium permittivity and conductivity receive new magnitudes ε_1 and σ_1. Further, we normalize all time variables to a frequency ω of the initial wave, $t \to \omega t$. This wave exists before the zero moment of time, the moment when the modulation commences, and is given by the function $E_0(t,x) = \exp[i(t-kx)]$. Each time jump of the medium properties changes the electromagnetic field, such that it is described by functions E_n on the disturbance interval and by F_n on the inactivity intervals.

After beginning of the modulation by the disturbance interval the initial wave is splitting into two, forward and backward, waves $E_1 = \exp\left(-st - ikx\right)\left[C_1\exp(iq\,t) + D_1\exp(-iq\,t)\right]$ with new amplitudes and new frequency $q = (a^2 - s^2)^{\frac{1}{2}}$ where $a^2 = \varepsilon/\varepsilon_1$, $s = \sigma_1/\omega\varepsilon_0\varepsilon$, and ε_0 is the vacuum permittivity. On the remaining undisturbed interval of this first modulation period the field splitting into two waves remains, $F_1 = \exp(-ik)\left[A_1\exp(i\,t) + B_1\exp(-i\,t)\right]$, but the frequency returns to the original one.

The field on the other disturbance intervals consists also of two, direct and inverse, waves $E_n = \exp(-st - ikx)\left[C_n\exp(iq\,t) + D_n\exp(-iq\,t)\right]$ of changed frequency while the field on the inactivity intervals consists of two waves $F_n = \exp(-ikx)\left[A_n\exp(i\,t) + B_n\exp(-i\,t)\right]$ but of the unchanged frequency. Therefore, the transformed field at any moment *t* of the *N*-th modulation period is given by the formula

$$E(t,x) = \sum_{n}^{N} E_n\theta(t-(n-1)T) - \sum_{n}^{N}\left\{E_n\theta(t-nT_1-(n-1)(T-T_1)) - F_n\theta(t-nT_1-(n-1)(T-T_1)) + F_n\theta(t-nT)\right\}. \qquad (2)$$

The expressions for the direct and the inverse secondary wave amplitudes are given in [3] where it is also shown that the relations between these amplitudes are determined by the ratios:

on the disturbance intervals $$w_N = \frac{D_N}{C_N}e^{-i2(N-1)qT} = \frac{\{p_2 + (p_1 - p_2)r_{N-1}\}\alpha_{21} + p_1p_2\alpha_{22}}{\{p_2 + (p_1 - p_2)r_{N-1}\}\varepsilon_{11} + p_1p_2\alpha_{12}}; \qquad (3)$$

on the inactivity intervals $$p_N = \frac{B_N}{A_N} e^{-i2NT} = \frac{p_1 p_2}{p_2 + (p_1 - p_2) r_N}, \; N \ge 2. \qquad (4)$$

Here, $p_1 = -h/m$, $p_2 = -h(m+m^*)/(hh^*+m^2)$, $A_1 = m\exp(-iT)$, $B_1 = -h\exp(iT)$, $\alpha_{11} = q+1+is$, $\alpha_{12} = q-1+is$, $\alpha_{21} = q-1-is$, $\alpha_{22} = q+1-is$, and the coefficients are introduced

$$m = \frac{1}{2q}\left[2q\cos(qT_1) + i(a^2+1)\sin(qT_1)\right]\exp[-sT_1 + i(T-T_1)] \qquad (5)$$

$$h = i\frac{1}{2q}\left(a^2 - 1 - i2s\right)\sin(qT_1)\exp[-sT_1 - i(T-T_1)] \qquad (6)$$

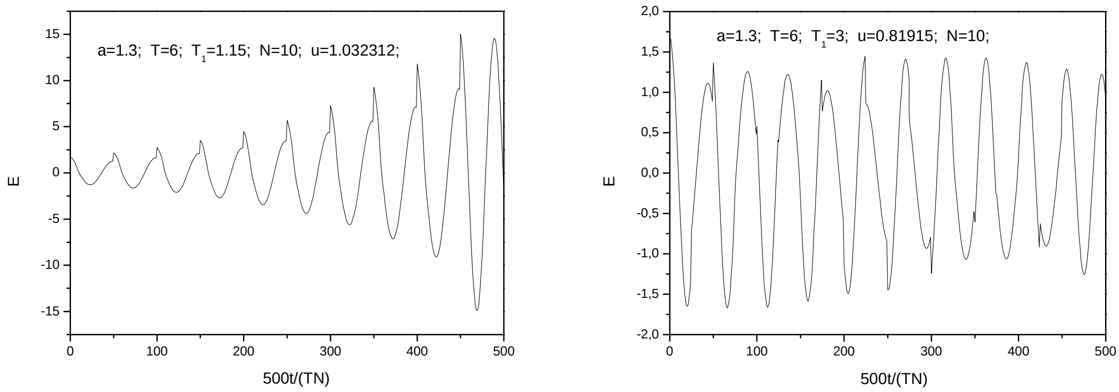
As it follows from (3) and (4) the behaviour of the ratios w_N and p_N between the amplitudes of the onward and the backward waves are governed by the sequence

$$r_{N+1} = 4u^2/(4u^2 - r_N), \qquad (7)$$

which is controlled by the generalized parameter

$$u = \cos(qT_1)\cos(T-T_1) - \frac{a^2+1}{2q}\sin(qT_1)\sin(T-T_1). \qquad (8)$$

If this parameter $u > 1$ then the sequence r_N has regular character and the transformed field undergoes a parametric amplification with time, Fig. 1a. Otherwise, when $u < 1$, the sequence as well as the field have irregular behaviour, Fig 1b, and the latter decreases as the medium possesses the dissipation.



a) b)

Fig. 1. The behaviour of the transformed field with time: a) parametric amplification, b) irregular changing.

3. QUASI-INTERMITTENCY AND HURST'S INDEX

The irregular behaviour of the field is seen more clearly in the ratios between the amplitudes of the forward and the backward waves, given by (3) and (4). These ratios are governed by the sequence r_n, which can have regular or irregular behaviour depending on the generalized parameter *u*. The sequence r_N behaves regularly if $u^2 > 1$. If $u^2 < 1$ the sequence r_N has irregular character that is there are long intervals in the sequence of the modulation periods where r_N changes almost regularly. After this interval the relatively short intervals of strong irregular behaviour of r_N takes place (upper Fig 2a). More distinct u^2 from 1 leads to more irregular behaviour of r_N (upper Fig. 2b). This phenomenon can be termed "quasi-intermittency" similar to [1].

The presence of the quasi-intermittency can be confirmed by the Hurst's method [3], according to which the time series of r_n is characterised by the Hurst's index H, determined by the asymptotic value of the function

$$H \sim \ln(R_n/S_n)/\ln n \qquad (9)$$

where

$$R_n = \max_{1\le k\le n} X(k,n) - \min_{1\le k\le n} X(k,n), \; X(k,n) = \sum_{i=1}^{k}\left(r_i - \langle r\rangle_n\right), \; \langle r\rangle_n = \frac{1}{n}\sum_{i=1}^{n} r_i, \quad S_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(r_i - \langle r\rangle_n\right)^2}. \qquad (10)$$

For the white noise (a completely uncorrelated signal) this index equals to $H = 0.5$. The value $H > 0.5$ $(H < 0.5)$ is associated with the long-range correlation when the time series exhibits persistence (antipersistence).

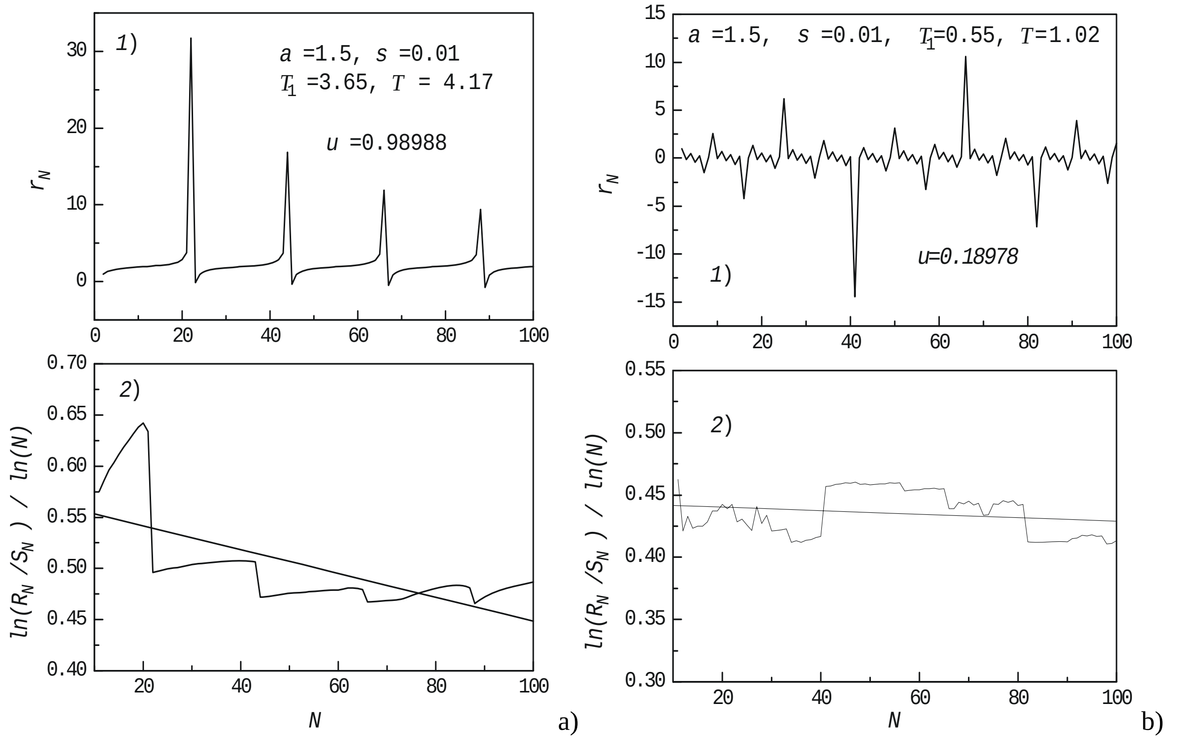


Fig. 2. Almost regular (a) and very irregular (b) behaviours of the sequence r_N and the corresponding Hurst's index.

The analysis of the sequence r_N in the considered case shows that the change of the amplitude ratio is associated with the uncorrelated process when $u^2 < 1$. For example, when $u \approx 0.98988$ the index H changes between ~0.55 and ~0.45, as illustrated by the lower Fig. 2. When u^2 decreases the behaviour becomes more irregular, see the lower Fig. 2b for $u = 0.18978$.

4. SIGNAL COMPLEXITY

The r_N behaviour can be also characterised by a complexity measure [2]. This measure of complexity shows how much information is stored in the signal and how much information is needed to predict the next value of the signal if we know all the values up to some moment in time. In two limiting cases, when a signal has constant value at all times and when the signal is completely random, a complexity is equal to zero in this framework because of no information about the previous evolution needed to predict the signal in both cases. All intermediate cases have a finite, non-zero value of a complexity.

The algorithm of computing the finite statistical complexity [4] follows the method originated in the works by Crutchfield and others and it consists of considering the symbolic subsequences that form the dynamical 'states' of the system and the time evolution, which is described as transitions between these states with some probabilities P_i. The finite statistical complexity is calculated by the formula:

$$C = -\sum_i P_i \log_2 P_i \qquad (11)$$

The dependence of this measure on the modulation period shows a correlation between the complexity and the generalized parameter *u*, Fig. 3. It is seen that the complexity drops to zero when the module of the parameter *u* (the dash-dot sine-like line) becomes greater than 1. In this case the value of the H index is typical for the regular behaviour. This is true for both cases when the medium becomes more or less optically dense on the disturbance intervals.

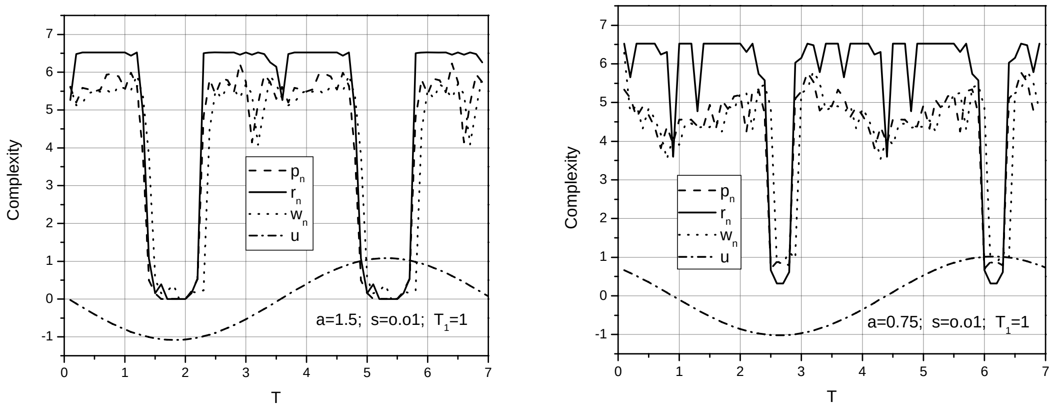
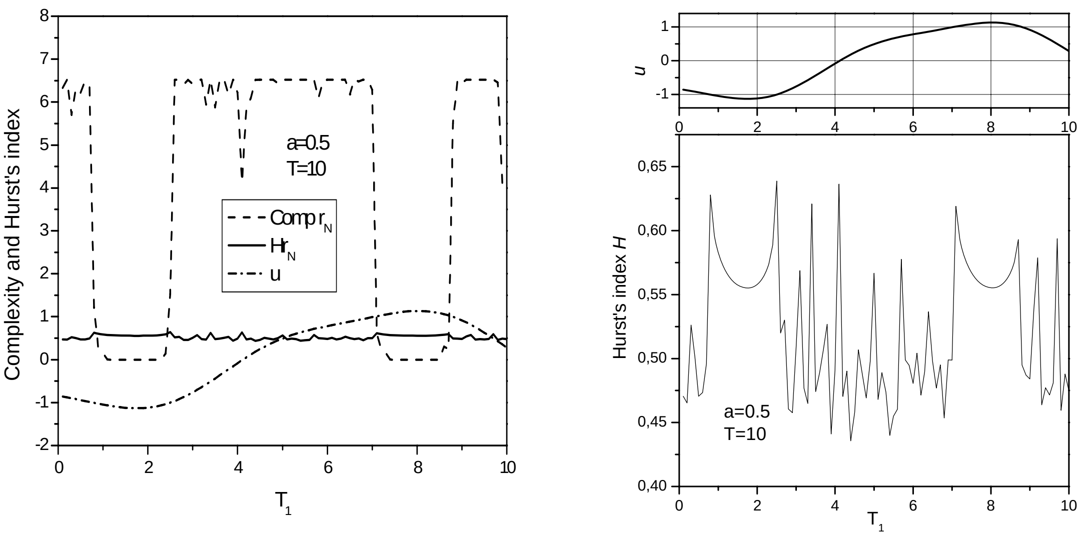


Fig. 3. The behaviour of the complexity vs the modulation periods for various changes of the permittivity (the parameter u is shown by the dash-dot sine-like line).

A correlation exists between the Hurst's index and the complexity of the signal. Fig. 4 shows the behaviour of these two characteristics for the sequence r_n depending on the duration of the disturbance interval. The complexity drops to zero when the Hurst's index deviates notably from the value 0.5 that corresponds to regular behaviour of the signal. Both characteristics correlate with the generalized parameter *u*.



a) b)

Fig. 4. The comparison (a) of Hurst's index H (solid line) and the complexity (dashed line) (the parameter u is the dash-dot line) and the detailed behaviour (b) of Hurst's index vs the duration of the disturbance interval.

5. CONCLUSION

The quasi-intermittency that occurs during the wave transformation under the time changes of the medium properties can be described by the two characteristics, the Hurst's index and the complexity measure. It is shown that these two characteristics correlate. They also correlate with the generalized parameter that controls the process of the wave transformation.

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