Time-varying Airy Pulses

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ABSTRACT
Pulses in the form of the Airy function as solutions to an equation similar to the Schrodinger equation but with opposite roles of the time and space variables are derived. The pulses are generated by an Airy time varying field at a source point and propagate in vacuum preserving their shape and magnitude. The pulse motion is decelerating according to a quadratic law. Its velocity changes from infinity at the source point to zero in infinity. These one dimensional results are extended to the 3D+time case for a similar Airy-Bessel pulse with the same behaviour, the non-diffractive preservation and the deceleration. This pulse is excited by the field at a plane aperture perpendicular to the direction of the pulse propagation.
Keywords: Airy pulses, decelerating electromagnetic pulses.

1. INTRODUCTION

Recently there has been active development in the theory and experimental applications of optical Airy beams [1-7]. An Airy beam or Airy wave packet is a wave described by the Airy function $\mathrm{y=Ai}[x]$, a special function which is the solution to the differential equation $y''-xy=0$ [8]. The Airy beams are characterised by very special properties: they are non-diffractive and accelerating. Among the striking applications of the optical Airy beams are the transport of small particles and living cells along a parabolic trajectory and the self-healing property of the beam, when the beam form is restored after passing an obstacle [9]. A new way of generating Airy beams by using three-wave mixing processes in nonlinear medium has been examined experimentally in [10].

The detailed analysis of the mathematical aspects as well as physical interpretation of electromagnetic Airy beams was done by considering the wave as a function of spatial coordinates only and assuming that their time dependence is harmonic, $\sim \exp(i\omega t)$, [2-8]. Yet, the idea of electromagnetic Airy beams comes from the analogy of the paraxial equation describing these beams with the time dependent Schrodinger equation [11], where the time variable is replaced with a spatial coordinate. It is worth to emphasize that the solution of the Schrodinger equation produces time dependent Airy wave packets in free space [11]. Their features such as the diffraction free form and continuous acceleration has been explained on the basis of the semi-classical approximation. (The Airy wave function is known in quantum mechanics for a long time [12] as a solution to the stationary Schrodinger equation.) As for the time dependent solution of the three-dimensional electromagnetic problem, the possibility of the existence of non-diffractive Bessel (not Airy) waves has been pointed out in [2, 13]. However, the three-dimensional solutions to the paraxial equations containing the time derivative do not include the parabolic variable responsible for the accelerating feature of the beams.

In this paper the explicitly time dependent solutions of the electromagnetic problem in the form of an Airy pulse are derived and investigated. We show that it is not only possible to find the Airy pulse solution starting from the first principles, rather than by exploiting the analogy with the paraxial equation, but also that the obtained pulse has the same property of non-diffractive propagation and velocity change without any external influences (in vacuum). There are, however, important conceptual differences that lead to the pulse deceleration, rather than acceleration as in quantum mechanics.

2. THE PARAXIAL TEMPORAL EQUATION

We start with the wave equation, followed from the Maxwell equation,

$$\partial^2_{zz}E(t,z)-c^{-2}\partial^2_{tt}E(t,z)=0\,, \tag{1}$$

which describes the electric field of a wave propagating along the z axis. Substitution of the field in the form $E(t,z)=B(t,z)e^{\pm ikz}$, $k=\omega/c$ and under the assumption that $\left|B''_{zz}\right| \ll \left|2ikB'_{z}\right|$, typical for the paraxial approximation [14-16], the wave equation is reduced to the form

$$\mp i2\partial_{\xi}B+\partial^2_{\tau\tau}B+\kappa^2 B=0\,, \tag{2}$$

where the normalised dimensionless variables are $\xi = z/(kc^2 t_0{}^2)$ and $\tau = t/t_0$ with t_0 being the temporal scale, $kct_0 = \kappa$ is the dimensionless parameter. Comparing this equation with the commonly considered spatial paraxial equation in the x ($s = x/x_0$, $x_0 = ct_0$), z (ξ) coordinates

$$i2\partial_\xi \Phi + \partial^2_{ss}\Phi = 0 \tag{3}$$

we see that the longitudinal spatial variables ξ are the same and the transverse variable s corresponds to the temporal variable τ in (2). The equation (3) is considered in the literature as the analogue to the Schrodinger equation [11]

$$\hbar^2 (2m)^{-1} \partial^2_{xx} \Psi(x,t) + i\hbar \partial_t \Psi(x,t) = 0 \tag{4}$$

from which the Airy wave packet originated in [2] if the temporal variable t in (4) is replaced by the longitudinal variable z (ξ in (3)). Thus, the variable z (ξ) along which an electromagnetic wave propagates plays the role of time in the electromagnetic phenomenon. As it was shown in [11] equation (4) has a solution in the form of a non-spreading wave packet with the envelope as the Airy function

$$\Psi(x,t) = \mathrm{Ai}\left[\frac{B}{\hbar^{2/3}}\left(x - \frac{B^3 t^2}{4m^2}\right)\right] exp\left[i\frac{B^3}{2m\hbar}\left(tx - \frac{B^3 t^3}{6m^2}\right)\right]. \tag{5}$$

This function describes the accelerating wave packet which moves uniformly with the velocity $\dot{x} = B^3 t / 2m^2$ and the constant acceleration $\ddot{x} = B^3 / 2m^2$. Contrary to the equation (3), which describes a beam harmonically oscillating in time, the function (5) represents the pulse with a complicated time varying envelope enclosed in the Airy function. The Airy function in the solution to (3)

$$\Phi = \mathrm{Ai}[s - \frac{1}{4}\xi^2]\exp\left\{i\left[-\frac{s\xi}{2} + \frac{1}{12}\xi^3\right]\right\} \tag{6}$$

describes the inhomogeneous distribution with respect to the spatial coordinates s and ξ of the wave paraxial propagating along the z axis but with harmonic temporal variation $E = \Phi(x,z)e^{ikz - i\omega t}$.

Our equation (2), derived from the first principle rather than by the analogy with the Schrodinger equation, shows that the roles of the time and space variables in the electromagnetic time paraxial equation (2) are opposite to those of the Schrodinger equation (4). This destroys the analogy between the equations (2) and (4) and, therefore, the direct correspondence between the time and space variables of the Schrodinger equation and the space variables of the spatial paraxial equation (3). Thus, we need to solve the equation (2) in order to find the time spatial pulse originating from it.

3. THE SOLUTION OF PARAXIAL TEMPORAL EQUATION

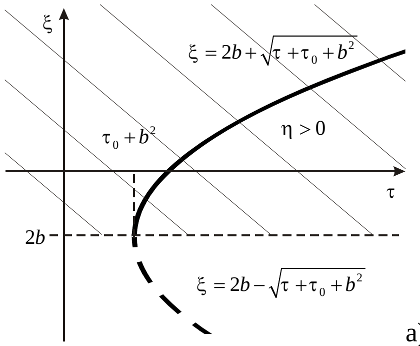
The solution to the equation (2) can be constructed following the procedure described in [7]. The sought function is represented as $B = W(\eta)e^{i\Theta(\eta,\xi)}$, where $W(\eta)$ and $\Theta(\eta,\xi)$ are real functions of the argument ξ and the quadratic variable $\eta = -a\tau + \tau_0 - g\xi^2/4 + b\xi$. The parameters a, τ_0, g, and b in η allow changing the model. The parameter $a = \pm 1$ determines the movement forward or backward along the spatial axes. The parameter g defines only the scale factor in variables normalization and can be taken $g = 1$. Substitution of $B = W(\eta)e^{i\Theta(\eta,\xi)}$ in (2) gives the Airy equation $W''(\eta) - \eta W(\eta) = 0$ with the solution as the Airy function W(η)=Ai[η]. Choosing $a = -1$ we obtain the solution to the equation (2) as

$$B(\tau,\xi) = \mathrm{Ai}\left[-\left(\frac{1}{2}\xi - b\right)^2 + \tau + \tau_0 + b^2\right]\exp\left\{\pm i\left[-(\tau + \tau_0 + b^2)\left(\frac{1}{2}\xi - b\right) + \frac{1}{12}\xi^3 - \frac{1}{2}b\xi^2 + \frac{(2b^2 - \kappa^2)}{2}\xi - b^3\right]\right\}. \tag{7}$$

Therefore, the solution (7) describes the field in the propagating pulse

$$E = \mathrm{Ai}\left[-\left(\frac{\xi}{2} - b\right)^2 + \tau + \tau_0 + b^2\right]\exp\left\{\pm i\left[-(\tau + \tau_0 + b^2)\left(\frac{\xi}{2} - b\right) + \frac{1}{12}\xi^3 - \frac{b}{2}\xi^2 + \frac{(2b^2 + \kappa^2)}{2}\xi - b^3\right]\right\}. \tag{8}$$

This field is uniquely defined in the half-space $\xi \geq 2b$, Fig.1a. Starting from the source point $\xi = 2b$, which can be interpreted as a time varying source and at which the field time variation is given by the formula $E(\tau, \xi = 2b) = \mathrm{Ai}\left[\tau + \tau_0 + b^2\right]\exp\left[\pm ib(\kappa^2 - b^2/3)\right]$, the field profile propagates according to the quadratic law $\tau + \tau_0 - (\xi/2 - b)^2 + b^2 = const$ preserving its form. Fig.1a illustrates the lines of propagation of the field equal values determined by the parabola $\tau+\tau_0-\xi^2/4+b\xi=const$ (one of the branches for $const=0$ is shown using the solid line in the figure).



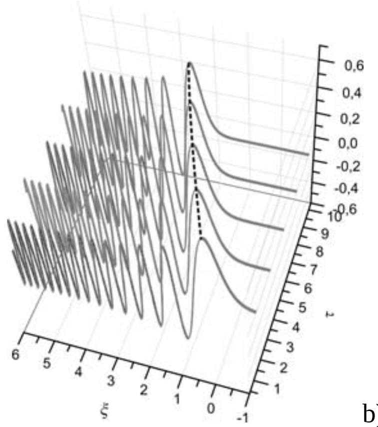


Figure 1. The pulse propagation: a) the region of definition of the electric field (hatched region); b) the propagation of the field equal values (the magnitudes of the envelope is shown on the vertical axes).

The quadratic variable η is positive inside the region bounded by the parabola and negative outside of it. It determines where the steep front of the Airy pulse is directed along the movement. The velocity of this movement decreases with distance $\dot{\xi}=2/(\xi-2b)$, therefore the acceleration $\ddot{\xi}=-4/(\xi-2b)^3$ is negative. Such a slowing motion leads to a complete stop as its velocity and acceleration tend to zero at the infinite distance from the source.

4. A FINITE ENERGY PULSE

The considered Airy pulses are of little practical importance because of their infinite energy. To overcome this deficiency it was suggested in [2, 3] to consider the exponentially decaying version at the input of the system. Following this suggestion we consider a different boundary condition

$$B(\tau,\xi=2b)=\mathrm{Ai}[\tau+\tau_0+b^2]\exp\left\{\pm i\left[-b(\kappa^2+b^2/3)\right]+\alpha(\tau+\tau_0+b^2)\right\} \quad (9)$$

for obtaining the pulse with finite energy. To solve the equation (2) with the boundary condition (9) we represent the solution via the Fourier transform $B(\tau,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu e^{i\nu\tau}\bar{B}(\nu,\xi)$. Then the boundary condition is $\bar{B}(\nu,2b)=\exp[i\nu(\tau_0+b^2)\mp ib(\nu^2+b^2/3)+i(\nu+i\alpha)^3/3]$. The equation for the inverse Fourier transform follows from (2)

$$\mp i2\partial_\xi\bar{B}(\nu,\xi)-(\nu^2-\kappa^2)\bar{B}(\nu,\xi)=0 \quad (10)$$

and its solution satisfying the boundary condition (9) determines the electric field of the pulse

$$E(\tau,\xi)=\mathrm{Ai}[u(\tau,\xi)]e^{iw(\tau,\xi)} \quad (11)$$

with

$$u(\tau,\xi)=\tau+\tau_0+b^2\mp 2i\alpha(\xi/2-b)-(\xi/2-b)^2, \quad (12)$$

$$w(\tau,\xi)=\exp i\left\{\pm b(\kappa^2-b^3/3)\pm\kappa^2(\xi/2-b)+2i\alpha(\xi/2-b)^2\pm 2(\xi/2-b)^3/3\mp[(\xi/2-b)\pm i\alpha][\tau+\tau_0+b^2]\right\}. \quad (13)$$

The steep front of the Airy envelope in (11), corresponding to nearly zero moment at the source point, comes off the source leaving the space free of the field.

5. 3D+TIME MODEL

The described model can be straightforwardly extended to the 3D+time case. If a phenomenon has the cylindrical symmetry then the wave equation in a cylindrical system of coordinates takes the form

$$\left(\partial^2_{\rho\rho}+\rho^{-1}\partial_\rho+\rho^{-2}\partial^2_{\varphi\varphi}+\partial^2_{zz}\right)E(t,\rho,\varphi,z)-c^{-2}\partial^2_{tt}E(t,\rho,\varphi,z)=0\,. \quad (14)$$

The solution to this equation in dimensionless variables is $E(\tau,r,\xi,\varphi)=J_m(\mu r)\mathrm{Ai}[u(\tau,\xi)]\exp[im\varphi+iw_1(\tau,\xi)]$ where $r=\rho/ct_0$ and $w_1(\tau,\xi)$ come from (13) by substituting $\kappa^2\to\kappa^2+\mu^2$. This solution describes the radiation from an aperture with the following distribution of the field

$$E(\tau,r,\xi=2b)=J_m(\mu r)\mathrm{Ai}[\tau+\tau_0+b^2]\exp\left\{\pm ib(\kappa^2+\mu^2-b^3/3)+\alpha[\tau+\tau_0+b^2]\right\}. \quad (15)$$

This radiation propagates normally to the aperture and slows down with distance to stopping.

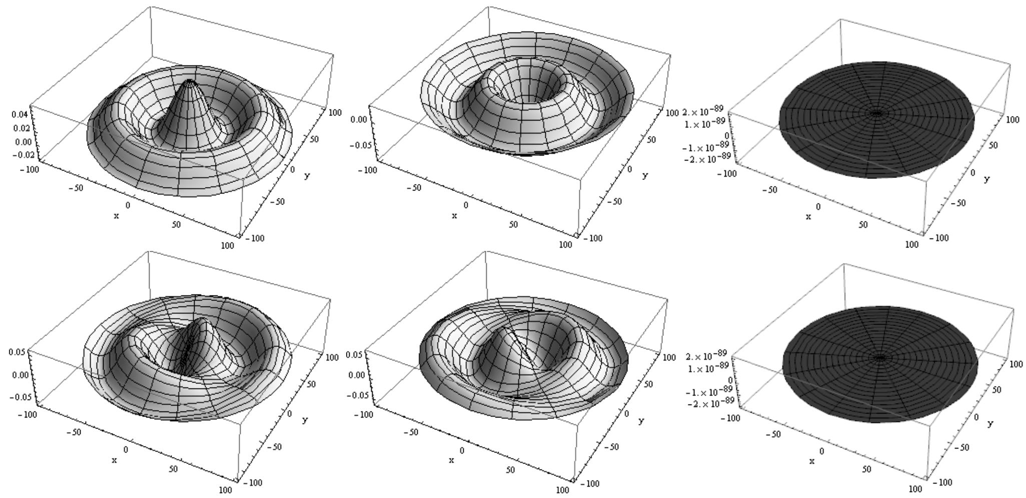


Figure 2. The evolution of the electric field at the point $\xi=10$ at three instants $\tau=0$ (left), $\tau=10$, and $\tau=100$ (right) for m=0 (top) and m=1 (bottom)

Fig. 2 illustrates the snapshots of the field transverse distribution at the point $\xi=10$ ($0\le r\le 100$, $0\le\varphi\le 2\pi$) at three instants. These distributions correspond to the radial symmetry (*m*=0, Fig. 2 top) and to one variation in the azimuthal angle (*m*=1, Fig. 2 bottom) at the aperture ($\mu=1$ in both cases). The snapshots at $\tau=0$ and $\tau=10$ show the pronounced inhomogeneous distributions of the field whereas at $\tau=100$ the field is essentially zero that means that the pulse comes off the source and leaves the space free of the field.

6. CONCLUSIONS

In conclusion, we derived the time dependent electromagnetic Airy pulses that satisfy the 'paraxial' equation similar to the Schrodinger equation in which the time and space variables interchange their roles. The solution to the electromagnetic equation is the Airy pulse which propagates with deceleration along its trajectory and stops at the infinite distance from the source. In the 3D case the similar Airy-Bessel pulse occurs when the radiation is excited by the field at the aperture which is perpendicular to the direction of the pulse propagation. If the field at the aperture is distributed as $J_m(\mu r)$ then the transverse distribution defined by the Bessel function propagates from the aperture preserving its form at all distances.

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